Irrationality exponents of certain fast converging series of rational numbers

Daniel Duverney, Takeshi Kurosawa, and Iekata Shiokawa

Abstract

Let $\{x_n\}$ be a sequence of rational numbers greater than one such that $x_{n+1} \geq x_n^2$ for all sufficiently large n and let $\varepsilon_n \in \{-1, 1\}$. Under certain growth conditions on the denominators of x_{n+1}/x_n^2 we prove that the irrationality exponent of the number $\sum_{n=1}^{\infty} \varepsilon_n/x_n$ is equal to $\limsup_{n\to\infty}(\log x_{n+1}/\log x_n)$.

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To the memory of Prof. Jonathan Sondow.

1 Introduction

Let $\{x_n\}_{n\geq 1}$ be a sequence of rational numbers greater than one such that $x_{n+1} \geq x_n^2$ for all sufficiently large n and let $\varepsilon_n \in \{-1, 1\}$. We consider the sum

$$S = \sum_{n=1}^{\infty} \frac{\varepsilon_n}{x_n}. \tag{1.1}$$

An easy induction shows that there exist constants $C > 0$ and $\rho \in (0, 1)$ such that

$$\left| \frac{\varepsilon_n}{x_n} \right| \leq C\rho^{2^n} \quad (n \geq 1).$$

Accordingly to [6], we call S a *fast converging series.* In this paper, we give in certain cases the exact value of the irrationality exponent of S, where the irrationality exponent $\mu(\alpha)$ of an irrational number α is defined by the supremum of the set of numbers μ for which the inequality

$$\left| \alpha - \frac{p}{q} \right| < \frac{1}{q^{\mu}}$$

has infinitely many rational solutions p/q. Every irrational α has $\mu(\alpha) \geq 2$ and, if $\mu(\alpha) > 2$, then α is transcendental by Roth's theorem.

For a non-zero rational number x, $\mathrm{den}(x)$ is defined by the smallest positive integer d for which dx is an integer. Our main result is stated as follows:

Theorem 1. *Let $\{x_n\}_{n\geq 1}$ be a sequence of rational numbers greater than one such that*

$$x_{n+1} \geq x_n^2 \qquad \text{for all sufficiently large } n \tag{1.2}$$

and let $\varepsilon_n \in \{-1, 1\}$. Define

$$\delta_1 = \operatorname{den} x_1, \quad \delta_{n+1} = \delta_n^2 \operatorname{den}\left(\frac{x_{n+1}}{x_n^2}\right) \qquad (n \geq 1). \tag{1.3}$$

Assume that

$$\log \delta_{n+1} = o(\log x_n) \tag{1.4}$$

as $n \to \infty$. Then the irrationality exponent of the number S defined in (1.1) is

$$\mu(S) = \limsup_{n\to\infty} \frac{\log x_{n+1}}{\log x_n}.$$

Corollary 1. *Let $\{A_n\}_{n\geq 1}$ be a strictly increasing sequence of positive integers and $\{B_n\}_{n\geq 1}$ be a sequence of non-zero integers such that $A_n/|B_n| > 1$ for all n. Set*

$$z_1 = A_1/|B_1|, \qquad z_{n+1} = \frac{A_{n+1}/|B_{n+1}|}{(A_n/B_n)^2} \quad (n \geq 1)$$

and define

$$\delta_1 = \operatorname{den} z_1, \qquad \delta_{n+1} = \delta_n^2 \operatorname{den} z_{n+1} \qquad (n \geq 1).$$

Assume that the following conditions are satisfied:

(i) *$z_{n+1} \geq 1$ for all sufficiently large n,*

(ii) *$\log |B_n| = o(\log A_n)$ as $n \to \infty$,*

(iii) *$\log \delta_{n+1} = o(\log A_n)$ as $n \to \infty$.*

Then

$$\mu\left(\sum_{n=1}^{\infty} \frac{B_n}{A_n}\right) = \limsup_{n\to\infty} \frac{\log A_{n+1}}{\log A_n}.$$

Amou and Bugeaud [1] proved a similar result with $z_{n+1} \geq 2$ for all sufficiently large n.

Remark 1. The assumption (1.4) implies that if $x_{n+1} = x_n^2$ for all sufficiently large n, then $x_n \in \mathsf{Z}_{>1}$ for every $n \geq 1$. Indeed, putting $N = \max\{n \geq 1 \mid x_n \neq x_{n-1}^2\}$ with $x_0 = 1$, we have $x_n = x_N^{2^{n-N}}$ and so $\delta_n = \delta_N^{2^{n-N}}$ $(n \geq N)$. Hence we get $2^{n+1-N} \log \delta_N = \log \delta_{n+1} = o(2^n)$ by (1.4), which implies that $\delta_N = 1$. Thus we have $\delta_n = 1$ $(n \geq 1)$. Similarly, if $z_n = 1$ for all sufficiently large n, then B_n divides A_n for every $n \geq 1$.

For the proof of Theorem 1, we express the sum S as a continued fraction in the case $x_1 > 2$ and $x_{n+1} \geq x_n^2$ for all $n \geq 2$ (see Section 2, Proposition 1). This expansion is essentially given in [1], and also in [11] when x_n are integers (see Section 4.3). Our proof is similar to that of Amou-Bugeaud [1], however, we must treat the continued fraction more carefully, because its partial quotients under the assumption (1.2) are non-negative rationals possibly less than one. So we study some of the properties of the continued fraction (see Section 2, Lemmas 2, 3, 4, and 5), which will be used in the proof of Theorem 1 given in Section 3. In the final Section 4, we give some applications of Theorem 1.

2 Continued fraction expansion of the series

We employ the standard notation for continued fractions:

$$[a_0; a_1, a_2, \dots] = \lim_{n\to\infty} [a_0; a_1, \dots, a_n],$$

where

$$[a_0; a_1, a_2, \dots, a_n] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}}.$$

Define p_n and q_n by

$$\begin{cases} p_{-1} = 1, & p_0 = a_0, \quad p_n = a_n p_{n-1} + p_{n-2}, \\ q_{-1} = 0, & q_0 = 1, \quad q_n = a_n q_{n-1} + q_{n-2}, \end{cases} \qquad (n \geq 1). \tag{2.1}$$

Then $[a_0; a_1, a_2, \dots, a_n] = p_n/q_n$, which is called the nth convergent. We use the formulas:

$$p_n q_{n-1} - p_{n-1} q_n = (-1)^{n+1}, \tag{2.2}$$

$$\frac{q_n}{q_{n-1}} = [a_n; a_{n-1}, \dots, a_2, a_1], \tag{2.3}$$

and also

$$[\dots, a, 0, b, \dots] = [\dots, a+b, \dots]. \tag{2.4}$$

Proposition 1. *Let $\{x_n\}_{n\geq 1}$ be a sequence of rational numbers such that*

$$z_1 = x_1 > 2, \quad z_{n+1} = x_{n+1} x_n^{-2} \geq 1 \qquad (n \geq 1) \tag{2.5}$$

and let $\varepsilon_n \in \{-1, 1\}$ with $\varepsilon_1 = 1$. Then the sums

$$S_n = \sum_{k=1}^{n} \frac{\varepsilon_k}{x_k}$$

have the following simple continued fraction expansions:

$$S_2 = \begin{cases} \left.\begin{array}{ll} [0; z_1 - 1, 1, z_2 - 1, z_1] & \text{if} \quad \varepsilon_2 = 1, \\ [0; z_1, z_2 - 1, 1, z_1 - 1] & \text{if} \quad \varepsilon_2 = -1 \end{array}\right\}, & z_2 \neq 1, \\ \left.\begin{array}{ll} [0; z_1 - 1, z_1 + 1] & \text{if} \quad \varepsilon_2 = 1, \\ [0; z_1 + 1, z_1 - 1] & \text{if} \quad \varepsilon_2 = -1 \end{array}\right\}, & z_2 = 1. \end{cases}$$

For $n \geq 2$, if

$$S_n = \left[0; a_1, a_2, \ldots, a_{l_n-1}, a^*_{l_n}\right] \tag{2.6}$$

with $a^*_{l_n} \neq 1$ is given, then writting

$$S_{n+1} = [0; a_1, a_2, \ldots, a_{l_n-1}, a_{l_n}, \ldots, a_{l_{n+1}}]$$

we have for $z_{n+1} \neq 1$

$$S_{n+1} = \begin{cases} \left[0, a_1, \ldots, a_{l_n-1}, a^*_{l_n}, z_{n+1}-1, 1, a^*_{l_n}-1, a_{l_n-1}, \ldots, a_1\right] & \text{if} \quad \varepsilon_{n+1} = 1, \\ \left[0, a_1, \ldots, a_{l_n-1}, a^*_{l_n}-1, 1, z_{n+1}-1, a^*_{l_n}, a_{l_n-1}, \ldots, a_1\right] & \text{if} \quad \varepsilon_{n+1} = -1 \end{cases} \tag{2.7}$$

and for $z_{n+1} = 1$

$$S_{n+1} = \begin{cases} \left[0, a_1, \ldots, a_{l_n-1}, a^*_{l_n}+1, a^*_{l_n}-1, a_{l_n-1}, \ldots, a_1\right] & \text{if} \quad \varepsilon_{n+1} = 1, \\ \left[0, a_1, \ldots, a_{l_n-1}, a^*_{l_n}-1, a^*_{l_n}+1, a_{l_n-1}, \ldots, a_1\right] & \text{if} \quad \varepsilon_{n+1} = -1. \end{cases} \tag{2.8}$$

Furthermore,

$$q^*_{l_n} = x_n \qquad (n \geq 2), \tag{2.9}$$

where $p^*_{l_n}$ and $q^*_{l_n}$ are rational numbers defined by (2.1) from $[0; a_1, \ldots, a_{l_n-1}, a^*_{l_n}]$, and

$$S = \lim_{n\to\infty} [0; a_1, a_2, \ldots, a_{l_n}].$$

Remark 2. If we denote the continued fraction expansion of S_{n+1} with $\varepsilon_{n+1} = 1$ in (2.7) or (2.8) as $\left[0; a_1, a_2, \ldots, a^*_{l_{n+1}}\right]$, then the expansion of S_{n+1} with $\varepsilon_{n+1} = -1$ is written by $\left[0; a^*_{l_{n+1}}, a_{l_{n+1}-1}, \ldots, a_2, a_1\right]$.

Example 1. The continued fraction expansions of S_3. Let $(\varepsilon_2, \varepsilon_3) = (1, 1)$.

$$\left.\begin{array}{ll} [0; z_1-1, 1, z_2-1, z_1, z_3-1, 1, z_1-1, z_2-1, 1, z_1-1] & \text{if} \quad z_2 \neq 1, \\ [0; z_1-1, z_1+1, z_3-1, 1, z_1, z_1-1] & \text{if} \quad z_2 = 1 \end{array}\right\}, \quad z_3 \neq 1,$$

$$\left.\begin{array}{ll} [0; z_1-1, 1, z_2-1, z_1+1, z_1-1, z_2-1, 1, z_1-1] & \text{if} \quad z_2 \neq 1, \\ [0; z_1-1, z_1+2, z_1, z_1-1] & \text{if} \quad z_2 = 1 \end{array}\right\}, \quad z_3 = 1.$$

Let $(\varepsilon_2, \varepsilon_3) = (-1, 1)$.

$$\left.\begin{array}{ll} [0; z_1, z_2-1, 1, z_1-1, z_3-1, 1, z_1-2, 1, z_2-1, z_1] & \text{if} \quad z_2 \neq 1, \\ [0; z_1+1, z_1-1, z_3-1, 1, z_1-2, z_1+1] & \text{if} \quad z_2 = 1 \end{array}\right\}, \quad z_3 \neq 1,$$

$$\left.\begin{array}{ll} [0; z_1, z_2-1, 1, z_1, z_1-2, 1, z_2-1, z_1] & \text{if} \quad z_2 \neq 1, \\ [0; z_1+1, z_1, z_1-2, z_1+1] & \text{if} \quad z_2 = 1 \end{array}\right\}, \quad z_3 = 1.$$

Remark 3. The length l_n of the continued fraction expansion of S_n given in Proposition 1 depends on the vanishing of $z_{n+1}-1$; namely, $l_{n+1} = 2l_n + 2$ if $z_{n+1} \neq 1, = 2l_n$ otherwise. The first $l_n - 1$ partial coefficients $a_1, a_2, \ldots, a_{l_n-1}$ of S_n and S_{n+1} coincide with each other. In the expansion of S_n, the last term $a^*_{l_n} = a_1$ for all $n \geq 3$ independently of ε's and z's. On the other hand, the l_nth partial denominator and succeeding few ones of the expansion of S_{n+1} depends on ε_{n+1} and the vanishing of $z_{n+1}-1$.

The key of the proof of Proposition 1 is the following:

Lemma 1 (cf. [1, Lemma F′], [12])**.** *Let $t, a_1, a_2, \ldots, a_k$ be positive real numbers and let $p_k/q_k = [0; a_1, a_2, \ldots, a_k]$. Assume that $a_k > 1$ and $t \geq 1$. Then*

$$\frac{p_k}{q_k} + \frac{(-1)^k}{tq_k^2} = [0; a_1, a_2, \ldots, a_k, t-1, 1, a_k - 1, a_{k-1}, \ldots, a_2, a_1],$$

$$\frac{p_k}{q_k} - \frac{(-1)^k}{tq_k^2} = [0; a_1, a_2, \ldots, a_{k-1}, a_k - 1, 1, t-1, a_k, \ldots, a_2, a_1].$$

Furthermore, we have $q_{2k+2} = tq_k^2$, where p_{2k+2}/q_{2k+2} is the $2k+2$th convergent representing each of the continued fractions in the right-hand sides.

Amou and Bugeaud assumed in Lemma F′ a slightly stronger condition that $a_j \geq 1$ $(1 \leq j \leq k)$, however the proof indicated there is valid also in the above cases $a_j > 0$ $(1 \leq j \leq k-1)$ and $a_k > 1$.

Proof of Proposition 1. The expansions of S_2 with $q^*_{l_2} = x_2$ can be obtained by direct calculation. Let $n \geq 2$ and S_n be given as in (2.6) with $a^*_{l_n} \neq 1$ and $q^*_{l_n} = x_n$. Assume that $z_{n+1} \neq 1$. In the case $\varepsilon_{n+1} = 1$, we apply Lemma 1 with $k = l_n$, $t = z_{n+1}$, and $q_k = q^*_{l_n}$ and get

$$\begin{aligned}
&[0; a_1, \ldots, a_{l_n - 1}, a^*_{l_n}, z_{n+1} - 1, 1, a^*_{l_n} - 1, a_{l_n - 1}, \ldots, a_1] \\
&= \frac{p^*_{l_n}}{q^*_{l_n}} + \frac{(-1)^{l_n}}{z_{n+1} {q^*_{l_n}}^2} = S_n + \frac{1}{x_{n+1}} = S_{n+1}
\end{aligned}$$

with $q^*_{l_{n+1}} = z_{n+1} q^*_{l_n} = z_{n+1} x_n^2 = x_{n+1}$. Similarly, we can prove (2.7) with $\varepsilon_{n+1} = -1$, as well as (2.8) by taking $t = z_{n+1} = 1$ in Lemma 1 and using (2.4). $\square$

Now we study some of the properties of the continued fraction $S = [0; a_1, a_2, a_3, \ldots]$. The next lemma can be easily deduced from Proposition 1 with Example 1.

Lemma 2. *Put*

$$A = \{1, z_1, z_1 \pm 1, z_1 \pm 2, z_2 - 1, z_3 - 1\} \setminus \{0\}.$$

Then $\{a_k \mid k \geq 1\} \subset A \cup \{z_j - 1 \mid j \geq 4\} \setminus \{0\}$. Furthermore, if a_k is of the form $z_j - 1$ for some $j \geq 4$, then $a_{k\pm1}, a_{k\pm2} \in A$.

Lemma 3. *The following inequalities hold:*

$$q_k > c_1 q_j \qquad (k > j \geq 1), \tag{2.10}$$

$$\alpha_k := [a_k; a_{k+1}, a_{k+2}, \ldots] > c_1 \qquad (k \geq 1), \tag{2.11}$$

where $c_1 = \min\{[0; a, b] \mid a, b \in A\} \in (0, 1)$.

Proof. We first prove that

$$q_k > c_1 q_{k-1} \qquad (k \geq 1). \tag{2.12}$$

If $a_k \in A$, then $q_k > a_k q_{k-1} \geq \min\{a \mid a \in A\} q_{k-1} > c_1 q_{k-1}$ noting that $a > [0; 1, a]$ for any $a \in A$. Otherwise, $a_{k-1}, a_{k-2} \in A$ by Lemma 2. Hence, we get $q_k/q_{k-1} > [0; a_{k-1}, a_{k-2}] \geq c_1$ by (2.3). Similarly, we can prove (2.11). Now, let $k > j \geq 1$. If $k - j$ is even, then $q_k > q_{k-2} > \cdots > q_j > c_1 q_j$. If $k - j \geq 3$ is odd, then $k - (j+1) \geq 2$ and is even. Hence, $q_k > q_{j+1} > c_1 q_j$ by (2.12). Thus, (2.10) is proved. □

Lemma 4. *For $n \geq 2$, we have*

$$q_k > c_1 x_n \qquad (l_n < k \leq l_{n+1}). \tag{2.13}$$

Proof. By (2.10), it is enough to show that $\max\{q_{l_n}, q_{l_n+1}\} \geq x_n$. Suppose that $z_{n+1} \neq 1$. Then, we have by (2.7) and (2.1)

$$q_{l_n} = q^*_{l_n} \text{ if } \varepsilon_{n+1} = 1, \tag{2.14}$$
$$q_{l_n+1} = 1 \cdot ((a^*_{l_n} - 1) q_{l_n-1} + q_{l_n-2}) + q_{l_n-1} = q^*_{l_n} \text{ if } \varepsilon_{n+1} = -1. \tag{2.15}$$

In the case $z_{n+1} = 1$, we have by (2.8) $q_{l_n} = (a^*_{l_n} + 1) q_{l_n-1} + q_{l_n-2} = q^*_{l_n} + q_{l_n-1}$ if $\varepsilon_{n+1} = 1$ and otherwise $q_{l_n+1} = (a^*_{l_n} + 1) q_{l_n} + q_{l_n-1} = (a^*_{l_n} + 1)((a^*_{l_n} - 1) q_{l_n-1} + q_{l_n-2}) + q_{l_n-1} > a^*_{l_n} q_{l_n-1} + q_{l_n-2} = q^*_{l_n}$. In any case, we find $\max\{q_{l_n}, q_{l_n+1}\} \geq x_n$ recalling (2.9). □

Lemma 5. *For $n \geq 2$, we have*

$$\delta_{n+1} p_k, \quad \delta_{n+1} q_k \in \mathsf{Z} \quad (l_n < k \leq l_{n+1}), \tag{2.16}$$

$$(\delta_{n+1} p_k, \delta_{n+1} q_k) \leq \delta^2_{n+1} \qquad (l_n < k \leq l_{n+1}). \tag{2.17}$$

Proof. Since p_k and q_k are sums of linear monomials of a_j $(1 \leq j \leq k)$ by (2.1), we have $p_k \prod_{j=1}^k \operatorname{den} a_j$, $q_k \prod_{j=1}^k \operatorname{den} a_j \in \mathsf{Z}$, where $\prod_{j=1}^k \operatorname{den} a_j \mid \prod_{j=1}^{l_{n+1}} \operatorname{den} a_j = \delta_{n+1}$ by Proposition 1, and hence (2.16) follows. The inequality (2.17) follows from (2.2). □

3 Proof of Theorem 1

For the proof of Theorem 1, we need the following lemma (cf., eg., [8], [10]):

Lemma 6. *If α is an irrational number, then*

$$\mu(\alpha) = \mu\left(\frac{a\alpha + b}{c\alpha + d}\right)$$

for any integers a, b, c, and d with $ad - bc \neq 0$.

Proof of Theorem 1. By Lemma 6, we may assume $x_1 > 2$. So we may expand the number S in the continued fraction as in Proposition 1. It follows from (2.1) and (2.2) that

$$\left| S - \frac{p_k}{q_k} \right| = \frac{1}{q_k(q_{k+1} + q_k/\alpha_{k+2})}, \tag{3.1}$$

where α_k is as in (2.11).

The proof will be divided into two cases. Case 1. $z_{n+1} \neq 1$ for infinitely many n. Case 2. $z_{n+1} = 1$ for all large n. Put for brevity

$$\tau = \limsup_{n\to\infty} \frac{\log x_{n+1}}{\log x_n}. \tag{3.2}$$

We note that $2 \leq \tau \leq \infty$ by (1.2).
Case 1. Now, we prove first that $\mu(S) \geq \tau$. Let $z_{n+1} \neq 1$ and let

$$l'_n = l_n \text{ if } \varepsilon_{n+1} = 1, \quad l'_n = l_n + 1 \text{ if } \varepsilon_{n+1} = -1.$$

By (2.14), (2.15), and (2.9), we have

$$q_{l'_n} = x_n, \quad a_{l'_n+1} = z_{n+1} - 1, \tag{3.3}$$

and $a_{l'_n+2}, a_{l'_n+3} \in A$ by (2.7) and Lemma 2. So $1/\alpha_{l'_n+2} > [0; a_{l'_n+2}, a_{l'_n+3}] \geq c_1$, where c_1 is as in Lemma 3. Hence we have by (2.3)

$$q_{l'_n+1} + q_{l'_n}/\alpha_{l'_n+2} > (z_{n+1} - 1 + c_1)q_{l'_n} \geq c_1 z_{n+1} q_{l'_n}. \tag{3.4}$$

For every positive integer n, define

$$r_n = \frac{\delta_{n+1}}{\left(\delta_{n+1}p_{l'_n}, \delta_{n+1}q_{l'_n}\right)}.$$

By Lemma 5, $r_n p_{l'_n}$ and $r_n q_{l'_n}$ are coprime integers and

$$\log r_n = O\left(\log \delta_{n+1}\right) = o\left(\log x_n\right).$$

It follows from (3.1) and (3.4) that

$$\left| S - \frac{r_n p_{l'_n}}{r_n q_{l'_n}} \right| = \left| S - \frac{p_{l'_n}}{q_{l'_n}} \right| < \frac{1}{c_1 z_{n+1} q_{l'_n}^2} = \frac{1}{c_1 \left(r_n q_{l'_n}\right)^{\sigma_n}}, \tag{3.5}$$

where σ_n is defined by

$$\sigma_n = 2 + \frac{\log z_{n+1} - 2\log r_n}{\log(r_n q_{l'_n})} = \frac{\log x_{n+1}}{\log x_n + \log r_n}.$$

We see that $\sigma_n \geq \tau - \varepsilon$ for ε arbitrarily small and n large. Now the set of the irreducible rational numbers $\{r_n p_{l'_n}/r_n q_{l'_n} \mid n \geq 1\}$ is infinite. Therefore S is irrational by (3.5) since $\tau \geq 2$ and $\mu(S) \geq \tau$. In particular, $\mu(S) = \infty$ if $\tau = \infty$. In what follows, we may assume that $\tau < \infty$.

Next, we prove that $\mu(S) \leq \tau$. Let $\varepsilon > 0$ be sufficiently small and let p/q be any reduced rational number with q sufficiently large. Assume first that there exists an integer k such that $p/q = p_k/q_k$ for some $k = k(q)$. Let n be such that $l'_n < k \leq l'_{n+1}$. We define

$$s_k = \frac{\delta_{n+1}}{(\delta_{n+1}p_k, \delta_{n+1}q_k)} \qquad \left(l'_n < k < l'_{n+1}\right),$$
$$s_{l'_{n+1}} = \frac{\delta_{n+2}}{\left(\delta_{n+2}p_{l'_{n+1}}, \delta_{n+2}q_{l'_{n+1}}\right)}.$$

By Lemma 5, $s_k p_k$ and $s_k q_k$ are coprime integers and therefore $p = s_k p_k$ and $q = s_k q_k$. Moreover

$$\log s_k = O\left(\log \delta_{n+1}\right) = o\left(\log x_n\right) \qquad \left(l'_n < k < l'_{n+1}\right),$$
$$\log s_{l'_{n+1}} = O\left(\log \delta_{n+2}\right) = o\left(\log x_{n+1}\right).$$

We have by (2.1), (2.10), and (2.11) for $l'_n < k \leq l'_{n+1}$

$$q_{k+1} + q_k/\alpha_{k+2} < (a_{k+1} + 2c_1^{-1})q_k \leq 2c_1^{-1}(a_{k+1}+1)q_k,$$

and so by (3.1)

$$\left|S - \frac{p}{q}\right| = \left|S - \frac{p_k}{q_k}\right| > \frac{c_1}{2(a_{k+1}+1)q_k^2} = \frac{c_1}{2q^{\tau_k}}, \tag{3.6}$$

where

$$\tau_k = \frac{2\log q_k + \log(a_{k+1}+1)}{\log q_k + \log s_k} = 2 + \frac{\log(a_{k+1}+1) - 2\log s_k}{\log q_k + \log s_k}. \tag{3.7}$$

If $k = l'_{n+1}$, then by (3.3) we can write for arbitrarily small ε and large n

$$\tau_{l'_{n+1}} = \frac{2\log x_{n+1} + \log z_{n+2}}{\log x_{n+1} + \log s_{l'_{n+1}}} \leq \left(1 + \frac{\varepsilon}{2\tau}\right)\left(2 + \frac{\log z_{n+2}}{\log x_{n+1}}\right)$$

and therefore

$$\tau_{l'_{n+1}} \leq \left(1 + \frac{\varepsilon}{2\tau}\right)\frac{\log x_{n+2}}{\log x_{n+1}} \leq \frac{\log x_{n+2}}{\log x_{n+1}} + \varepsilon \leq \tau + 2\varepsilon. \tag{3.8}$$

Assume that $l'_n < k < l'_{n+1}$. By (3.2), there exists a positive integer $n_0 = n_0(\varepsilon)$ such that

$$\log z_{n+1} < (\tau - 2 + \varepsilon)\log x_n \qquad (n \geq n_0). \tag{3.9}$$

Since $a_{k+1} + 1 \leq \max\{z_1 + 3, z_2, z_3, \ldots, z_{n+1}\}$ by Lemma 2, we get for n sufficiently large

$$\begin{aligned}\log(a_{k+1}+1) &\leq \max\left\{\log(z_1+3), \max_{2\leq j\leq n_0}\log z_j, (\tau - 2 + \varepsilon)\max_{n_0\leq j\leq n}\log x_j\right\} \\ &\leq (\tau - 2 + \varepsilon)\log x_n\end{aligned}$$

using (3.9). Therefore by (3.7) and Lemma 4

$$\tau_k \le 2 + \frac{(\tau - 2 + \varepsilon)\log x_n - 2\log s_k}{\log x_n + \log c_1 + \log s_k} \le \tau + 2\varepsilon. \tag{3.10}$$

Hence by (3.6), (3.8), and (3.10), we have

$$\left|S - \frac{p}{q}\right| > \frac{1}{q^{\tau+2\varepsilon}} \tag{3.11}$$

for all $p/q \in \{p_k/q_k \mid k \ge 1\}$ with q sufficiently large.

Next, assume that $p/q \notin \{p_j/q_j \mid j \ge 1\}$. There exist k such that $q_k^{1-\varepsilon} \le 2q < q_{k+1}^{1-\varepsilon}$. Let n such that $l_n < k \le l_{n+1}$. By Lemma 4, we have $\delta_{n+1} < q_k^{\varepsilon}$, and so

$$\delta_{n+1} q_k q < \frac{1}{2} q_k^{1+\varepsilon} q_{k+1}^{1-\varepsilon} < \frac{1}{2c_1^{\varepsilon}} q_k q_{k+1} < \frac{2}{3} q_k q_{k+1}$$

noting that $q_k/q_{k+1} < 1/c_1$ by (2.12). Thus using (3.1) we have

$$\begin{aligned}\left|S - \frac{p}{q}\right| &\ge \left|\frac{p}{q} - \frac{\delta_{n+1}p_k}{\delta_{n+1}q_k}\right| - \left|S - \frac{p_k}{q_k}\right| > \frac{1}{\delta_{n+1}q_k q} - \frac{1}{q_k q_{k+1}} \\ &> \frac{1}{3\delta_{n+1}q_k q} > \frac{1}{3q_k^{1+\varepsilon} q} > \frac{1}{q^{2+4\varepsilon}},\end{aligned}$$

for all $p/q \notin \{p_j/q_j \mid j \ge 1\}$ with q sufficiently large, which together with (3.11) implies $\mu(S) \le \tau$, and therefore, $\mu(S) = \tau$.
Case 2. Let $z_{n+1} = 1$ for all large n. Then we have $\tau = 2$ and $x_n \in \mathsf{Z}_{>1}$ for every $n \ge 1$ by Remark 1. So it is enough to prove that

$$\mu\left(\sum_{n=1}^{\infty} \frac{\varepsilon_n}{x_1^{2^{n-1}}}\right) = 2,$$

which was shown already by Derevyanko [5] (see also Sondow [13, Corollary 3]), and the proof of Theorem 1 is completed. □

Proof of Corollary 1. Put $x_n = A_n/|B_n|$ and $\varepsilon_n = \operatorname{sgn} B_n$ $(n \ge 1)$. Then the assumptions (i) and (iii) with (ii) lead to (1.2) and (1.4), respectively. Hence, we can apply Theorem 1 getting

$$\mu\left(\sum_{n=1}^{\infty} \frac{\varepsilon_n}{x_n}\right) = \limsup_{n\to\infty} \frac{\log A_{n+1} - \log|B_{n+1}|}{\log A_n - \log|B_n|} = \limsup_{n\to\infty} \frac{\log A_{n+1}}{\log A_n}.$$

□

4 Applications

In this section, we exhibit some examples obtained as applications of Theorem 1. We first remark that, in computing the irrationality exponents of a series, the first few terms are negligible by Lemma 6, namely,

$$\mu\left(\sum_{n=1}^{\infty}\frac{\varepsilon_n}{x_n}\right)=\mu\left(\sum_{n=n_0}^{\infty}\frac{\varepsilon_n}{x_n}\right)$$

holds for every fixed $n_0 \geq 1$.

4.1 Gap series

Example 2. For any integers a and b with $1 \leq a \leq b$ and $b \geq 2$, we define the sequence $\{x_n\}$ by $x_1 = a$ and for $2^k \leq n < 2^{k+1}$ with $k \geq 1$

$$x_n = a^{2^{n-1}}\left(\frac{b}{a}\right)^{2^{n-2}+2^{n-2^2}+2^{n-2^3}+\cdots+2^{n-2^k}} \in \mathsf{Z}_{>0}.$$

Then, $z_1 = a$ and

$$z_n = \begin{cases} b/a & \text{if} \quad n = 2^k \quad \text{for some} \quad k \geq 1 \\ 1 & \text{otherwise} \end{cases},$$

and by Theorem 1

$$\mu\left(\sum_{n=1}^{\infty}\frac{\varepsilon_n}{x_n}\right)=2.$$

In particular, if $a = b$, then we find $x_n = a^{2^{n-1}}$ and $z_n = 1$ for all $n \geq 1$, i.e., Derevyanko's case stated above.

Example 3. For integers $a \geq 1$ and $b > 1$, we have

$$\mu\left(\sum_{n=1}^{\infty}\varepsilon_n\frac{a^{2^n}}{b^{3^n}}\right)=3.$$

4.2 Engel series and Pierce series

Engel series and Pierce series are series of the forms

$$\sum_{n=1}^{\infty}\frac{1}{q_1q_2\cdots q_n} \qquad \text{and} \qquad \sum_{n=1}^{\infty}\frac{(-1)^n}{q_1q_2\cdots q_n}$$

respectively, where $\{q_n\}_{n\geq 1}$ is a non-decreasing sequence integers such that $q_1 \geq 2$. As an immediate consequence of Theorem 1, we have

Corollary 2. *Let $\{q_n\}_{n\geq 1}$ be a sequence of positive integers. Assume that $q_1 \geq 2$ and*

$$q_1q_2\cdots q_n \,|q_{n+1} \qquad (n \geq 1)\,.$$

Then

$$\mu\left(\sum_{n=1}^{\infty}\frac{\varepsilon_n}{q_1q_2\cdots q_n}\right)=1+\limsup_{n\to\infty}\frac{\log q_{n+1}}{\log\left(q_1q_2\cdots q_n\right)}.$$

4.3 Hone's reciprocal sums

Let $\{x_n\}$ be a sequence of positive integers such that

$$x_1 = 1, \quad x_2 \geq 2, \quad x_n^2 \mid x_{n+1}, \quad x_{n+1} \geq 2x_n^2. \tag{4.1}$$

Hone [11] expanded the sum

$$S = \sum_{n=1}^{\infty} \frac{1}{x_n}$$

in the continued fraction and proved its transcendence. As an example, he took up the sequence $\{x_n\}$ generated by the rational recurrence

$$x_{n+2}x_n = x_{n+1}^M F(x_{n+1}) \tag{4.2}$$

from the initial values $x_0 = x_1 = 1$, where

$$M \geq 3, \quad F(x) \in \mathsf{Z}_{\geq 0}[x], \quad d = \deg F \geq 1, \quad F(0) \neq 0.$$

It is easily seen that $\{x_n\}$ is a sequence of positive integers satisfying the condition (4.1). Hone proved that

$$\log x_n = c_2\lambda^n + O(1), \tag{4.3}$$

where $c_2 > 0$ is a constant and

$$\lambda = \frac{M + d + \sqrt{(M+d)^2 - 4}}{2} \geq \frac{3 + \sqrt{5}}{2} > 2.6$$

is one of the roots of the equation $\lambda^2 - (M+d)\lambda + 1 = 0$.

Corollary 3. *Let $\{x_n\}$ be the sequence defined by (4.2) and let a be a positive integer. Then we have*

$$\mu\left(\sum_{n=1}^{\infty} \varepsilon_n \frac{a^n}{x_n}\right) = \lambda.$$

4.4 Cahen's constant

Sylvester's sequence $\{S_n\}$ is defined by the nonlinear recurrence

$$S_0 = 2, \quad S_{n+1} = S_n^2 - S_n + 1 \ \ (n \geq 0).$$

Cahen [3] proved the irrationality of the number

$$C = \sum_{n=0}^{\infty} \frac{(-1)^n}{S_n - 1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{x_n} = 0.64341\ldots,$$

where $x_n = S_n - 1$ satisfy the recurrence

$$x_0 = 1, \quad x_{n+1} = x_n(x_n + 1) \ \ (n \geq 0). \tag{4.4}$$

Davison and Shallit [4] established the transcendence of Cahen's constant C via its continued fraction expansion and Becker [2] improved their result by a variant of Mahler's method. Finch [9, Section 6.7] asked the arithmetical properties of the number
$$\sum_{n=0}^{\infty} \frac{1}{S_n - 1} = 1.691030\cdots.$$
Recently, the authors [7, Example 1.5] proved that, for any algebraic numbers $a \neq 0$ and $\gamma \neq S_n$ $(n \geq 0)$ and any positive integers l, the number
$$\sum_{n=0}^{\infty} \frac{a^n}{(S_n - \gamma)^l}$$
is transcendental except when $a = l = 1$ and $\gamma = 0$, in which case
$$\sum_{n=0}^{\infty} \frac{1}{S_n} = 1.$$

We consider the sequence $\{x_n\}$ of positive integers generated by
$$x_0 = 1, \quad x_{n+1} = x_n^m F(x_n) \qquad (n \geq 0), \tag{4.5}$$
where
$$m \geq 2, \quad F(x) \in \mathsf{Z}_{\geq 0}[x], \quad d = \deg F \geq 1, \quad F(0) \neq 0. \tag{4.6}$$

Lemma 7. *Let $\{x_n\}$ be the sequence defined by (4.5) with (4.6). Then*
$$\log x_n = c(m+d)^n + O(1), \tag{4.7}$$
where $c > 0$ is a constant.

Proof. By (4.5) and (4.6) we have
$$\log x_k = (m+d) \log x_{k-1} + \log \left(c_0 + \frac{c_1}{x_{k-1}} + \cdots + \frac{c_d}{x_{k-1}^d} \right),$$
where $c_k \geq 0$ with $c_0 c_d \neq 0$ are the coefficients of $F(x)$. Multiplying both sides by $(m+d)^{n-k}$ and summing up from $k = 2$ to n yields
$$\begin{aligned} \log x_n &= (m+d)^{n-1} \log x_1 \\ &\quad + (m+d)^n \sum_{k=2}^{n} \frac{1}{(m+d)^k} \log \left(c_0 + \frac{c_1}{x_{k-1}} + \cdots + \frac{c_d}{x_{k-1}^d} \right), \end{aligned}$$
where the last sum converges as $n \to \infty$, since $x_n \geq 1$ and $m + d \geq 3$. Thus we can write
$$\begin{aligned} &\log x_n \\ &= (m+d)^n \left(\frac{\log x_1}{m+d} + \sum_{k=2}^{\infty} \frac{1}{(m+d)^k} \log \left(c_0 + \frac{c_1}{x_{k-1}} + \cdots + \frac{c_d}{x_{k-1}^d} \right) \right) \\ &\quad - \sum_{k=n+1}^{\infty} \frac{1}{(m+d)^{k-n}} \log \left(c_0 + \frac{c_1}{x_{k-1}} + \cdots + \frac{c_d}{x_{k-1}^d} \right), \end{aligned}$$

which leads to (4.7). □

Corollary 4. *Let $\{x_n\}$ be the sequence defined by (4.5) and let a be a positive integer. Then we have*

$$\mu\left(\sum_{n=0}^{\infty}\varepsilon_n\frac{a^n}{x_n}\right)=m+d.$$

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DANIEL DUVERNEY:
Baggio Engineering School, Lille, France
e-mail: daniel.duverney@orange.fr

TAKESHI KUROSAWA:
Department of Applied Mathematics, Tokyo University of Science, Tokyo, Japan
e-mail:tkuro@rs.tus.ac.jp

IEKATA SHIOKAWA:
Department of Mathematics, Keio University, Yokohama, Japan
e-mail: shiokawa@beige.ocn.ne.jp