ARITHMETICAL FUNCTIONS AND IRRATIONALITY OF LAMBERT SERIES

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Abstract : We use a method of Erdös in order to prove the linear independence over $\mathbb{Q}$ of the numbers

$$1, \ \sum_{n=1}^{+\infty} \frac{1}{q^{n^2}-1}, \ \sum_{n=1}^{+\infty} \frac{n}{q^{n^2}-1}$$

for every $q \in \mathbb{Z}$, with $|q| \geq 2$. The main idea consists in considering the two above series as Lambert series. This allows to expand them as power series of $1/q$. The Taylor coefficients of these expansions are arithmetical functions, whose properties allow to apply an elementary irrationality criterion, which yields the result.

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1. Introduction

Let q be a rational integer satisfying $|q| \geq 2$. Define

$$\alpha = \sum_{n=1}^{+\infty} \frac{1}{q^{n^2}-1}, \ \beta = \sum_{n=1}^{+\infty} \frac{n}{q^{n^2}-1} \tag{1}$$

The aim of this paper is to prove the following theorem.

Theorem 1 : *Let q be a rational integer satisfying $|q| \geq 2$. Then the numbers 1, α and β are linearly independent over $\mathbb{Q}$.*

This theorem generalizes a result of Paul Erdös, who proved in 1948 [7] the irrationality of the following Lambert series:

$$\gamma = \sum_{n=1}^{+\infty} \frac{1}{q^n-1} = \sum_{n=1}^{+\infty} \frac{d(n)}{q^n}. \tag{2}$$

Here $d(n)$ is the divisor function, defined by

$$d(n) = \sum_{x|n} 1 \tag{3}$$

By a very precise study of the properties of the divisor function, Erdös succeeded in showing that the q-adic expansion of γ is not ultimately periodic for $q \geq 2$, which implies the irrationality of γ.

The pattern of the proof of theorem 1 will be basically the same. In fact, α is also a Lambert series (see [8] , page 257, or [6], page 102, for the general properties of Lambert series). To show it, first observe that

$$\alpha = \sum_{x=1}^{+\infty} \frac{1}{q^{x^2} - 1} = \sum_{x=1}^{+\infty} \frac{1}{q^{x^2}} \frac{1}{1 - \dfrac{1}{q^{x^2}}} = \sum_{x=1}^{+\infty}\sum_{r=1}^{+\infty} \frac{1}{q^{rx^2}}.$$

Now, if we reorder the double series by putting $n = rx^2$, we see that

$$\alpha = \sum_{n=1}^{+\infty} \frac{a(n)}{q^n}, \tag{4}$$

where the arithmetical function $a(n)$ is defined by

$$a(n) = \sum_{x^2 |\, n} 1. \tag{5}$$

Similarly, we have

$$\beta = \sum_{n=1}^{+\infty} \frac{b(n)}{q^n}, \tag{6}$$

where

$$b(n) = \sum_{x^2 |\, n} x. \tag{7}$$

Following the ideas of Erdös, the proof of theorem 1 will use an elementary criterion of irrationality, very similar to those used in [2] , [4] and [5], which will be proved in section 2. Then we will study in detail, in section 3, the arithmetical functions $a(n)$ and $b(n)$. Theorem 1 will be proved in section 4.

It should be noted that Erdös' result in [7] has been generalized by Peter Borwein [1] in a very different way. By an explicit computation of the Padé approximants of the fonction

$$L_q(x) = \sum_{n=1}^{+\infty} \frac{x}{q^n - x} = \sum_{n=1}^{+\infty} \frac{x^n}{q^n - 1}, \tag{8}$$

Borwein proved that $L_q(x)$ is irrational for every rational integer q such that $|q| \geq 2$ and every $x \in \mathbb{Q}^*$, $|x| < |q|$.

Padé approximants allow in fact to prove that $L_q(x)$ is irrational for every non zero rational x, $|x| < |q|$, and every rational $q = r/s$ satisfying

$$\frac{\operatorname{Log}|s|}{\operatorname{Log}|r|} < \frac{1}{4}\left(3 - \sqrt{5 + \frac{24}{\pi^2}}\right).$$

See [3] for details. Moreover, the use of Padé approximants can be extended to irrationality proofs for series whose general term involves second order recurring sequences. For example, Matala-Aho and Prevost have proved in [9] that

$$\delta = \sum_{n=1}^{+\infty} \frac{x^n}{F_n} \tag{9}$$

is irrational for every $x \in \mathbb{Q}^*$, $|x| < \dfrac{1+\sqrt{5}}{2}$. Here F_n denotes the Fibonacci sequence.

However, this powerful approach seems difficult to use in the case where q^n is replaced by q^{n^2}, because it rests heavily on the fact that the function L_q satisfies a functional equation, namely

$$L_q(qx) = L_q(x) + \frac{x}{1-x}. \tag{10}$$

There is no such functional equation if we replace q^n by q^{n^2} in L_q.

Arithmetic sequences of integers will play an important part in the proof of theorem 1. We recall here an elementary lemma on their divisibility properties. For non zero natural integers A, B, d and n, denote

$$E_{A,B,d}(n) = \{i \in \{1, 2, ..., n\} \ / \ d \text{ divides } Ai + B\}.$$

Lemma 1 : *If A and B are coprime, then $E_{A,B,d}(n)$ has at most $\left[\dfrac{n}{d}\right] + 1$ elements.*

Proof : If d divides $Ai + B$, then A and d are coprime since A and B are coprime. Hence the equation $Ai + B \equiv 0 \pmod{d}$ has exactly one solution i_0 satisfying $0 \leq i_0 \leq d - 1$.
Then for every solution i of this equation, we have $A(i - i_0) \equiv 0 \pmod{d}$ and d divides $i - i_0$. Hence the solutions i are exactly all the numbers of the form $jd + i_0$, with $j \in \mathbb{Z}$.
The condition $1 \leq jd + i_0 \leq n$ leads to $0 \leq j \leq \left[\dfrac{n}{d}\right]$.

This proves that $E_{A,B,d}(n)$ has at most $\left[\frac{n}{d}\right]+1$ elements.

2. An irrationality criterion

Theorem 2 : *Let $q \in \mathbb{Z}$, $|q| \geq 2$. Let $[\theta(n)]_{n\in\mathbb{N}}$ be a sequence of rational integers. Assume that there exist a sequence of natural integers $(n_k)_{k\in\mathbb{N}}$ with $n_k \geq 2k$ such that, for every k sufficiently large,*

$$\left\{ \begin{array}{l} q \mid \theta(n_k-k+1)\,,\ q^2 \mid \theta(n_k-k+2)\,,\ \cdots\,,\ q^{k-1} \mid \theta(n_k-1)\,, \\ q^{k+1} \mid \theta(n_k+1)\,,\ q^{k+2} \mid \theta(n_k+2)\,,\ \cdots\,,\ q^{2k} \mid \theta(n_k+k)\,, \end{array} \right. \tag{11}$$

and satisfying

$$\lim_{k\to+\infty} \frac{1}{|q|^k} \sum_{n=0}^{+\infty} \frac{|\theta(n+n_k+k+1)|}{|q|^n} = 0. \tag{12}$$

Assume that $\displaystyle\sum_{n=0}^{+\infty} \frac{\theta(n)}{q^n}$ is convergent and is a rational number.
Then $q^k \mid \theta(n_k)$ for every large k.

Proof : If $\displaystyle\sum_{n=0}^{+\infty} \frac{\theta(n)}{q^n} = \frac{\lambda}{\mu}$, where λ and μ are rational integers, then for every k we have

$$\sum_{n=0}^{n_k-k} \frac{\theta(n)}{q^n} + \sum_{n=n_k-k+1}^{n_k-1} \frac{\theta(n)}{q^n} + \frac{\theta(n_k)}{q^{n_k}} + \sum_{n=n_k+1}^{n_k+k} \frac{\theta(n)}{q^n} + \sum_{n=n_k+k+1}^{+\infty} \frac{\theta(n)}{q^n} = \frac{\lambda}{\mu}.$$

By (11), there exists an integer A_k such that

$$\mu \left[\sum_{n=0}^{n_k-k} \frac{\theta(n)}{q^n} + \frac{A_k}{q^{n_k-k}} + \frac{\theta(n_k)}{q^{n_k}} \right] - \lambda = -\mu \sum_{n=n_k+k+1}^{+\infty} \frac{\theta(n)}{q^n}.$$

If we multiply this equality by q^{n_k}, we see that there exists an integer B_k such that

$$\mu \left[B_k q^k + \theta(n_k) \right] - \lambda q^{n_k} = -\mu \frac{1}{q^{k+1}} \sum_{n=0}^{+\infty} \frac{\theta(n+n_k+k+1)}{q^n}.$$

From (11) we deduce that the integer $\mu\left[B_k q^k + \theta(n_k)\right] - \lambda q^{n_k}$ is equal to zero for every large k. As $n_k \geq 2k$, this implies that q^k divides $B_k q^k + \theta(n_k)$ for every large k, which proves theorem 2.

3. The arithmetical functions $a(n)$ and $b(n)$

Lemma 2 : *The arithmetical function $a(n)$ is multiplicative. Moreover, if $n = \prod_i p_i^{\alpha_i}$, where the p_i are distinct primes, then*

$$a(n) = \prod_i \left(\left[\frac{\alpha_i}{2} \right] + 1 \right) \tag{13}$$

Proof : First we observe that $a(n)$ is multiplicative. Indeed, if r and s are coprime, then every integer x such that x^2 divides rs is of the form $x = x_1 x_2$, where x_1^2 divides r and x_2^2 divides s. Thus, if r and s are coprime,

$$a(rs) = \sum_{x_1^2 | r} \sum_{x_2^2 | s} 1 = \sum_{x_1^2 | r} 1 \sum_{x_2^2 | s} 1 = a(r)a(s). \tag{14}$$

Therefore we have to compute $a(n)$ only when $n = p^\alpha$, where p is prime. But in this case the integers x such that x^2 divides p^α are exactly 1, p, p^2, $\cdots$, $p^{\left[\frac{\alpha}{2}\right]}$, which proves lemma 2.

Lemma 3 : *The arithmetical function $b(n)$ is multiplicative. Moreover, if $n = \prod_i p_i^{\alpha_i}$, where the p_i are distinct primes, then*

$$b(n) = \prod_i \left(1 + p + p^2 + \cdots + p^{\left[\frac{\alpha_i}{2}\right]} \right). \tag{15}$$

Proof : Similar to the proof of lemma 2.

Lemma 4 : *Let $q \in \mathbb{Z}$, $|q| \geq 2$. Let p be any prime satisfying $p > |q|$. Then there exists a natural integer $\omega = \omega(p)$ satisfying $\omega \leq 2q^2$, such that q divides $a(p^\omega)$ and $b(p^\omega)$, .*

Proof : As $\mathbb{Z}/q\mathbb{Z}$ has $|q|$ elements, at least two of the $|q| + 1$ numbers 1, $1 + p$, $1 + p^2$, ..., $1 + p + ... + p^{|q|}$ are equal modulo q. Therefore there exist two integers i and j, with $1 \leq i < j \leq |q|$, such that

$$q \mid p^i \left(1 + p + ... + p^{j-i}\right).$$

As $|q| < p$, this yields $q \mid 1 + p + ... + p^{j-i}$. Now

$$\sum_{k=0}^{|q|(j-i+1)-1} p^k = \frac{p^{|q|(j-i+1)} - 1}{p - 1} = \frac{p^{j-i+1} - 1}{p - 1} \sum_{k=0}^{|q|-1} p^{k(j-i+1)}.$$

Hence $q \mid \sum_{k=0}^{|q|(j-i+1)-1} p^k$. Therefore, if we put $\omega = 2\left[|q|(j - i + 1) - 1\right]$, we see by lemmas 2 and 3 that $q \mid a(p^\omega)$ and $q \mid b(p^\omega)$.
Moreover, $\omega \leq 2\left[|q|(|q| - 1)\right] \leq 2q^2$, which proves lemma 4.

Lemma 5 : *Assume that A and B are two coprime natural integers. Assume that $n \geq 8\sqrt{An + B}$. Then the set S of the numbers $Ai + B$, for $i = 1, 2, ...,$*

n contains at least $\left[\frac{n}{8}\right]$ squarefree numbers. If ζ is one of these numbers, then clearly $a(\zeta) = b(\zeta) = 1$.

Proof : Let p be any prime number.
If $p > \sqrt{An+B}$, then p^2 cannot divide any $Ai+B$, since $i \leq n$.
If $p \leq \sqrt{An+B}$, then by lemma 1 there exist at most $\left[\frac{n}{p^2}\right] + 1$ numbers $Ai+B$ such that p^2 divides $Ai+B$.
Hence the number of squarefree numbers in S is at least

$$\eta \geq n - \sum_{\substack{p \text{ prime} \\ p \leq \sqrt{An+B}}} \left(\left[\frac{n}{p^2}\right] + 1 \right) \geq n - n \sum_{p \text{ prime}} \frac{1}{p^2} - \sqrt{An+B}.$$

But we have $\sum_{p \text{ prime}} \frac{1}{p^2} \leq \sum_{r=2}^{+\infty} \frac{1}{r^2} \leq \frac{1}{4} + \int_2^{+\infty} \frac{du}{u^2} = \frac{3}{4}$, whence

$$\eta \geq n - \frac{3}{4}n - \frac{1}{8}n \geq \frac{n}{8}.$$

Lemma 6 : *Let $n \in \mathbb{N}$ be sufficiently large, and let A and B be two coprime natural integers such that A is odd and $A \leq \sqrt{n}$. Then the set S of the numbers $Ai+B$, for $i = 1, 2, ..., n$ contains at least $\left[\frac{n}{80}\right]$ numbers ζ such that $a(\zeta) = 2$* and *$b(\zeta) = 3$.*

Proof : As A is odd and A and B are coprime, there exists B' such that $8B' + 4 \equiv B \pmod{A}$, $2B'+1$ and A are coprime, and $1 \leq B' \leq A$. Clearly we have, for every $k \in \mathbb{N}$, $2Ak + 2B' + 1 \leq A(2k+3)$.
Hence the numbers

$$h_k = 4(2Ak + 2B' + 1) = 8Ak + 8B' + 4$$

belong to S for $k = 1, 2, ..., \left[\frac{n}{9}\right]$ and n sufficiently large, since

$$h_k \leq 4A\left(\frac{2n}{9} + 3\right) \leq An + B$$

for n sufficiently large.
By lemma 5, the set T of the numbers $2Ak + 2B' + 1$ contains at least $\left[\frac{1}{8}\left[\frac{n}{9}\right]\right]$ squarefree numbers, and these numbers are odd. Hence the sequence h_k contains at least $\left[\frac{1}{8}\left[\frac{n}{9}\right]\right] \geq \left[\frac{n}{80}\right]$ numbers ζ with no other square divisors than 1 and 4, that is satisfying $a(\zeta) = 2$ and $b(\zeta) = 1 + 2 = 3$.

Lemma 7 : *Let n, A and B be non zero natural integers such that A and B are coprime and $\max(A, B) \leq \sqrt{n}$. Then*

$$\sum_{i=1}^{[\frac{n}{A}]} a\left(Ai+B\right) \leq 4\frac{n}{A}. \qquad (16)$$

Proof : Denote $\mathbb{N}^0 = \mathbb{N} - \{0\}$. We have

$$\sum_{i=1}^{[\frac{n}{A}]} a\left(Ai+B\right) = \sum_{i=1}^{[\frac{n}{A}]} \sum_{x^2|\ Ai+B} 1 \leq \sum_{(x,y)\in D} 1,$$

where $D = \left\{(x,y) \in \mathbb{N}^0 \times \mathbb{N}^0 \ / \ x^2y \leq n+B \ ; \ x^2y \equiv B \pmod{A}\right\}$.
Hence by lemma 1 we can write

$$\begin{aligned}\sum_{i=1}^{[\frac{n}{A}]} a\left(Ai+B\right) &\leq \sum_{x=1}^{[\sqrt{n+B}]} \left(\left[\frac{n}{Ax^2}\right]+1\right) \leq \frac{n}{A}\sum_{x=1}^{[\sqrt{n+B}]} \frac{1}{x^2} + \sqrt{n+B} \\ &\leq 2\frac{n}{A} + \frac{n+\sqrt{n}}{\sqrt{n}} \leq \frac{2n}{A} + \frac{2n}{A} = \frac{4n}{A}.\end{aligned}$$

Lemma 8 : *Let n, A and B be non zero natural integers such that A and B are coprime, $n \geq 3$ and $\max(A,B) \leq \sqrt{n}$. Then*

$$\sum_{i=1}^{[\frac{n}{A}]} b\left(Ai+B\right) \leq 20\frac{n\,\mathrm{Log}\,n}{A} \qquad (17)$$

Proof : As in lemma 7, we have

$$\sum_{i=1}^{[\frac{n}{A}]} b\left(Ai+B\right) = \sum_{i=1}^{[\frac{n}{A}]} \sum_{x^2|\ Ai+B} x \leq \sum_{(x,y)\in D} x,$$

where $D = \left\{(x,y) \in \mathbb{N}^0 \times \mathbb{N}^0 \ / \ x^2y \leq n+B \ ; \ x^2y \equiv B \pmod{A}\right\}$. We define

$$\begin{aligned} D_1 &= \left\{(x,y) \in D \ / \ x \leq (n+B)^{\frac{1}{4}}\right\}, \\ D_2 &= \left\{(x,y) \in D \ / \ y \leq (n+B)^{\frac{1}{2}}\right\}. \end{aligned}$$

It is easy to see that $D \subset (D_1 \cup D_2)$, whence

$$\sum_{i=1}^{[\frac{n}{A}]} b\left(Ai+B\right) \leq \sum_{(x,y)\in D_1} x + \sum_{(x,y)\in D_2} x. \qquad (18)$$

We put now $s = \left[(n+B)^{\frac{1}{4}}\right]$.

a) First we look for an upper bound for $\sum_{(x,y)\in D_1} x$.

Recall that $n \geq 3$ and $\max(A,B) \leq \sqrt{n}$. We have, as in the proof of lemma 7,

$$\begin{aligned}\sum_{(x,y)\in D_1} x &\leq \sum_{x=1}^{s}\left(\left[\frac{n}{Ax^2}\right]+1\right)x \leq \frac{n}{A}\sum_{x=1}^{s}\frac{1}{x}+\sum_{x=1}^{s}x \\ &\leq \frac{n}{A}(\operatorname{Log} s+1)+s^2 \leq \frac{n}{A}\left(\frac{1}{4}\operatorname{Log}(n+B)+1\right)+\sqrt{n+B} \\ &\leq \frac{n}{4A}(\operatorname{Log} n+\operatorname{Log} 2+4)+\frac{n+B}{\sqrt{n+B}} \leq \frac{n}{4A}\times 6\operatorname{Log} n+\frac{2n}{A} \leq 4\frac{n\operatorname{Log} n}{A}.\end{aligned}$$

b) Now we look for an upper bound for $\sum_{(x,y)\in D_2} x$. We have

$$\sum_{(x,y)\in D_2} x = \sum_{y=1}^{\left[\sqrt{n+B}\right]} \sum_{\substack{x^2y\leq n+B \\ x^2y\equiv B \pmod{A}}} x.$$

For a given y, the congruence $x^2y \equiv B \pmod{A}$ has at most two solutions $x_{1,y}$ and $x_{2,y}$ satisfying $0 < x_{1,y} < A$, $0 < x_{2,y} < A$ and $x_1 \neq x_2$. As A and B are coprime, the solutions of this congruence are exactly the $x = Ar + x_{q,y}(y)$ for some $q \in \{1,2\}$. Therefore

$$\begin{aligned}\sum_{(x,y)\in D_2} x &\leq \sum_{y=1}^{\left[\sqrt{n+B}\right]} \sum_{q=1}^{2} \sum_{(Ar+x_{q,y})^2 y\leq n+B} (Ar+x_{q,y}) \\ &\leq 2A \sum_{y=1}^{\left[\sqrt{n+B}\right]} \sum_{r=0}^{\left[\frac{1}{A}\sqrt{\frac{n+B}{y}}\right]} (r+1).\end{aligned}$$

As $n(n+1) \leq 4(n-1)^2$ for $n \geq 3$, we obtain

$$\sum_{(x,y)\in D_2} x \leq \frac{4}{A}\sum_{y=1}^{\left[\sqrt{n+B}\right]} \frac{n+B}{y} \leq \frac{4n}{A}(\operatorname{Log} n+\operatorname{Log} 2+2) \leq 16\frac{n\operatorname{Log} n}{A}.$$

This completes the proof of lemma 8.

4. Proof of theorem 1

Assume that α and β are linearly dependent over $\mathbb{Q}$. Then there exist two rational integers λ and μ such that $(\lambda,\mu) \neq (0,0)$ and $\lambda\alpha + \mu\beta \in \mathbb{Q}$. This means that

$$\sum_{n=1}^{+\infty} \frac{\lambda a(n)+\mu b(n)}{q^n} \in \mathbb{Q}. \tag{19}$$

We will show that this is impossible by using theorem 2 with

$$\theta(n) = \lambda a(n) + \mu b(n). \tag{20}$$

In all the proof, k is a natural integer sufficiently large.
For every $i \geq 1$, we denote

$$t_i = \frac{i(i+1)}{2},\ r_i = t_i + 1, \tag{21}$$

where t_i is the i^{th} triangular number. We recall for further use that, for every $i \geq 1$,

$$t_{i+1} - t_i = i + 1. \tag{22}$$

We denote by C_1, C_2, ..., positive real numbers which may depend on q, λ or μ, but not on k, and we put $\varepsilon = 0$ or 1.

Step 1 : Let $p_1, p_2, p_3, \ldots$ be the series of the successive prime numbers greater than k^{20}. For every prime p, let $\omega = \omega(p)$ be defined in lemma 4. We will use the Chinese Remainder Theorem. First, there exists a natural number η_k such that

$$\begin{cases} \eta_k - k + 1 \equiv p_1^{\omega(p_1)} & \left(\bmod p_1^{\omega(p_1)+1}\right) \\ \eta_k - k + 2 \equiv p_2^{\omega(p_2)} p_3^{\omega(p_3)} & \left(\bmod p_2^{\omega(p_2)+1} p_3^{\omega(p_3)+1}\right) \\ \vdots & \vdots \\ \eta_k - 1 \equiv p_{r_{k-2}}^{\omega\left(p_{r_{k-2}}\right)} \cdots p_{t_{k-1}}^{\omega\left(p_{t_{k-1}}\right)} & \left(\bmod p_{r_{k-2}}^{\omega\left(p_{r_{k-2}}\right)+1} \cdots p_{t_{k-1}}^{\omega\left(p_{t_{k-1}}\right)+1}\right) \\ \eta_k + 1 \equiv p_{r_k}^{\omega\left(p_{r_k}\right)} \cdots p_{t_{k+1}}^{\omega\left(p_{t_{k+1}}\right)} & \left(\bmod p_{r_k}^{\omega\left(p_{r_k}\right)+1} \cdots p_{t_{k+1}}^{\omega\left(p_{t_{k+1}}\right)+1}\right) \\ \eta_k + 2 \equiv p_{r_{k+1}}^{\omega\left(p_{r_{k+1}}\right)} \cdots p_{t_{k+2}}^{\omega\left(p_{t_{k+2}}\right)} & \left(\bmod p_{r_{k+1}}^{\omega\left(p_{r_{k+1}}\right)+1} \cdots p_{t_{k+2}}^{\omega\left(p_{t_{k+2}}\right)+1}\right) \\ \vdots & \vdots \\ \eta_k + k \equiv p_{r_{2k-1}}^{\omega\left(p_{r_{2k-1}}\right)} \cdots p_{t_{2k}}^{\omega\left(p_{t_{2k}}\right)} & \left(\bmod p_{r_{2k-1}}^{\omega\left(p_{r_{2k-1}}\right)+1} \cdots p_{t_{2k}}^{\omega\left(p_{t_{2k}}\right)+1}\right) \end{cases} \tag{23}$$

As the arithmetical functions a and b are multiplicative, we know by lemma 4 and (20) that (11) is satisfied, with n_k replaced by η_k. Moreover, if we define

$$A_k = \prod_{i=1}^{t_{k-1}} p_i^{\omega(p_i)+1} \prod_{i=r_k}^{t_{2k}} p_i^{\omega(p_i)+1} \tag{24}$$

we know by the Chinese Remainder Theorem that we can choose η_k satisfying

$$0 \leq \eta_k \leq A_k. \tag{25}$$

Now we look for an upper bound for A_k. Let $p'_1 = 2$, $p'_2 = 3$, $p'_3 = 5$, ... be the series of all prime numbers. The elementary Chebyshev inequality

$$\pi(n) \geq n \operatorname{Log} 2 / \operatorname{Log} n$$

immediatly yields

$$p'_n \leq C_1 n \operatorname{Log} n. \tag{26}$$

Therefore, as $p_i \geq k^{20}$, we have by lemma 4

$$\begin{aligned} A_k &\leq \prod_{i=1}^{t_{2k}} \left(p'_{i+k^{20}}\right)^{2q^2+1} \leq \prod_{i=1}^{t_{2k}} \left[C_1 \left(i+k^{20}\right) \operatorname{Log}\left(i+k^{20}\right)\right]^{2q^2+1} \\ &\leq \prod_{i=1}^{t_{2k}} \left[2C_1 k^{20} \operatorname{Log}\left(2k^{20}\right)\right]^{2q^2+1} \leq \left[2C_1 k^{20} \operatorname{Log}\left(2k^{20}\right)\right]^{\left(q^2+1\right)k(2k+1)}. \end{aligned}$$

Hence, for k sufficiently large, we have

$$\eta_k \leq A_k \leq \exp\left(k^{2.5}\right). \tag{27}$$

Step 2 : Put $N_k = \left[\dfrac{2^{k^{10}}}{A_k}\right]$. We consider now all the numbers of the form

$$u_{k,i} = iA_k + \eta_k,\ i = 1, 2, \cdots, N_k. \tag{28}$$

It is clear that every $u_{k,i}$ satisfies the system of congruences (23), as η_k does. Consequently, we have

$$\left\{ \begin{array}{l} q \mid \theta(u_{k,i}-k+1),\ q^2 \mid \theta(u_{k,i}-k+2),\ \cdots,\ q^{k-1} \mid \theta(u_{k,i}-1), \\ q^{k+1} \mid \theta(u_{k,i}+1),\ q^{k+2} \mid \theta(u_{k,i}+2),\ \cdots,\ q^{2k} \mid \theta(u_{k,i}+k). \end{array} \right. \tag{29}$$

Recall that $\varepsilon = 0$ or 1. We define

$$\mathbb{E}_k = \{i \in \mathbb{N} \ / \ 1 \leq i \leq N_k \ / \ a(u_{k,i}) = 1+\varepsilon, b(u_{k,i}) = 1+2\varepsilon\}. \tag{30}$$

As (27) holds, we can apply lemmas 5 and 6, and we see that the cardinal of $\mathbb{E}_k$ satisfies

$$|\mathbb{E}_k| \geq \left[\frac{N_k}{80}\right] \geq \frac{2^{k^{10}}}{81A_k} \tag{31}$$

Now we look for an upper bound of the sum

$$S_k = \sum_{i\in\mathbb{E}_k} \sum_{n=0}^{10k^{10}} |\theta(n+u_{k,i}+k+1)|. \tag{32}$$

We clearly have (33):

$$\begin{aligned} S_k &\leq \sum_{i=1}^{N_k} \sum_{n=0}^{10k^{10}} \left(|\lambda|\, a\left(n + iA_k + \eta_k + k + 1\right) + |\mu|\, b\left(n + iA_k + \eta_k + k + 1\right)\right). \\ &\leq \sum_{n=0}^{10k^{10}} \sum_{i=1}^{N_k} \left(|\lambda|\, a\left(iA_k + n + \eta_k + k + 1\right) + |\mu|\, b\left(iA_k + n + \eta_k + k + 1\right)\right). \end{aligned}$$

Now we show that A_k and $n + \eta_k + k + 1$ are coprime. Indeed, if not, by (24) it would exist some p_j such that $p_j \mid n + \eta_k + k + 1$, with $1 \leq j \leq t_{k-1}$ or $r_k \leq j \leq t_{2k}$. But p_j divides $\eta_k + k - g$ for some g between 0 and $2k - 1$ and $g \neq k$ by (23). Therefore p_j would divide $n + g + 1$, which is impossible because $p_j \geq k^{20}$ and $n + g + 1 \leq 10k^{10} + 2k$.
Hence A_k and $n + \eta_k + k + 1$ are coprime. By lemmas 7 and 8 we get

$$S_k \leq \sum_{n=0}^{10k^{10}} \left(4\,|\lambda|\, \frac{2^{k^{10}}}{A_k} + 20 \,\mathrm{Log}\, 2 \frac{k^{10} 2^{k^{10}}}{A_k}\, |\mu| \right) \leq C_2 \frac{k^{20} 2^{k^{10}}}{A_k}. \qquad (34)$$

Denote $m_k = \min_{i \in \mathbb{E}_k} \left(\sum_{n=0}^{10k^{10}} |\theta\left(n + u_{k,i} + k + 1\right)| \right)$.

By (31), (32) and (34) we have $m_k \dfrac{2^{k^{10}}}{81 A_k} \leq C_2 \dfrac{k^{20} 2^{k^{10}}}{A_k}$.
Therefore $m_k \leq k^{21}$ for every k sufficiently large.
Hence there exists $i_k \in \{1, 2, \cdots, N_k\}$ such that

$$\sum_{n=0}^{10k^{10}} |\theta\left(n + u_{k,i_k} + k + 1\right)| \leq k^{21}. \qquad (35)$$

Define $n_k = u_{k,i_k}$. By (29), (30) and (35), we have

$$\begin{cases} q \mid \theta\left(n_k - k + 1\right),\ q^2 \mid \theta\left(n_k - k + 2\right),\ \cdots,\ q^{k-1} \mid \theta\left(n_k - 1\right), \\ q^{k+1} \mid \theta\left(n_k + 1\right),\ q^{k+2} \mid \theta\left(n_k + 2\right),\ \cdots,\ q^{2k} \mid \theta\left(n_k + k\right), \\ \sum_{n=0}^{10k^{10}} |\theta\left(n + n_k + k + 1\right)| \leq k^{21}. \\ a(n_k) = 1 + \varepsilon, b\left(n_k\right) = 1 + 2\varepsilon \end{cases} \qquad (36)$$

Step 3 : We show now that theorem 2 applies with n_k defined above. We have only to check that (12) holds, since

$$n_k \geq A_k \geq p_1 \geq k^{20} \geq 2k.$$

We have by (36)

$$\begin{aligned}\sum_{n=0}^{+\infty}\frac{|\theta(n+n_k+k+1)|}{|q|^n} &\leq k^{21}+\sum_{n=10k^{10}}^{+\infty}\frac{|\theta(n+n_k+k+1)|}{|q|^n}\\ &\leq k^{21}+C_3\sum_{n=10k^{10}}^{+\infty}\frac{(n+n_k+k+1)^2}{|q|^n}.\end{aligned}$$

Now $10k^{10}+n_k+k+1\leq\exp(k^{10})$ by (27) and (28). Hence

$$\sum_{n=0}^{+\infty}\frac{|\theta(n+n_k+k+1)|}{|q|^n}\leq k^{21}+\frac{C_3}{|q|^{10k^{10}}}\sum_{n=0}^{+\infty}\frac{\left(n+\exp(k^{10})\right)^2}{|q|^n}\leq 2k^{21}.$$

This proves that (12) holds. Theorem 2 applies and

$$q^k\mid\lambda a(n_k)+\mu b(n_k) \tag{37}$$

for every large k. Taking successively $\varepsilon=0$ and $\varepsilon=1$ in (36), we obtain

$$\begin{cases} q^k\mid\lambda+\mu\\ q^k\mid 2\lambda+3\mu\end{cases}$$

Hence q^k divides λ and μ for every large k, which yields $\lambda=\mu=0$ and proves theorem 1.

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