On series involving Fibonacci and Lucas numbers I

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1 Introduction

This paper is the first of two survey papers devoted to the study of series involving Fibonacci and Lucas numbers. There has been, up to now, a great deal of works on this subject, and it seems necessary to summarize and classify it.

To begin with, we observe that series involving Fibonacci and Lucas numbers can be divided, roughly, into two large classes ; first, the subscript of Fibonacci and Lucas numbers appear in arithmetic progression ; second, they appear in geometric progression.

We give here two classical examples of both cases. For these two exmples, the sum of the series can be explicitly computed.

Example 1.1. We have

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{F_n F_{n+1}} = \Psi = -\frac{1}{\Phi}, \tag{1.1}$$

where $\Phi = \dfrac{1+\sqrt{5}}{2}$ is the Golden Number.

It seems difficult to find the first mention of this result, which is a direct consequence of the well-known continued fraction expansion of Ψ (see for example [15], Exercise 3.11). But it is undoubtedly very old.

Example 1.2. Lucas found in 1878 [27] that

$$\sum_{n=0}^{\infty} \frac{1}{F_{2^n}} = \frac{7-\sqrt{5}}{2}. \tag{1.2}$$

This result can be easily obtained by taking $x = \Psi$ in de Morgan's series (cf. [9])

$$\sum_{n=0}^{\infty} \frac{x^{2^n}}{1-x^{2^{n+1}}} = \frac{x}{1-x}, \tag{1.3}$$

which is readily deduced from the identity

$$\frac{x}{1-x^2} = \frac{x}{1-x} - \frac{x^2}{1-x^2}. \tag{1.4}$$

There are three natural questions about series involving Fibonacci and Lucas numbers.

First question: Given a series, can we compute its sum? That means, can we express this series in a closed form as in Examples 1.1 and 1.2, or alternatively, can we express it by using classical functions? These will include, as will see later, q-hypergeometric functions, Lambert series, elliptic integrals, and theta functions.

Second question: Given a series, can we study its arithmetical properties? First of all, can we prove this series to be irrational? For example, we know that $\sum_{n=0}^{\infty} \frac{1}{F_n}$ is irrational (cf. [2],[10],[13]).

Third question: Given a series, can we prove it to be transcendental? This last problem is the most difficult. However, a number of results have been obtained recently on the subject. For example, it is known that $\sum_{n=0}^{\infty} \frac{1}{F_{2n+1}}$ is transcendental (cf. [17], but it is not known if $\sum_{n=0}^{\infty} \frac{1}{F_n}$ is transcendental!)

In this first paper, we will give a survey of the answers to the first question. The other paper, with the same title but numbered II, will be devoted to the answers to the second and third questions.

2 Computation in closed form when the subscripts are in arithmetic progression

2.1 Products in the denominators

A very elementary result is

$$\sum_{n=1}^{\infty} \frac{1}{F_n F_{n+2}} = 1. \tag{2.1}$$

Althought this series looks like (1.1), its summation is quite different and much more simple. Observe that

$$\frac{1}{F_n F_{n+2}} = \frac{1}{F_{n+1}} \left(\frac{1}{F_n} - \frac{1}{F_{n+2}} \right) = \frac{1}{F_n F_{n+1}} - \frac{1}{F_{n+1} F_{n+2}},$$

so that (2.1) is a telescoping series.

Now let us define for every $k \geq 0$

$$S_k = \sum_{n=1}^{\infty} \frac{1}{F_n F_{n+1} \cdots F_{n+k}}, \quad S_k^* = \sum_{n=1}^{\infty} \frac{(-1)^n}{F_n F_{n+1} \cdots F_{n+k}}, \tag{2.2}$$

$$T_k = \sum_{n=1}^{\infty} \frac{1}{F_n F_{n+k}}, \quad T_k^* = \sum_{n=1}^{\infty} \frac{(-1)^n}{F_n F_{n+k}}. \tag{2.3}$$

In paticular,

$$S_0 = \sum_{n=1}^{\infty} \frac{1}{F_n}, \quad S_0^* = \sum_{n=1}^{\infty} \frac{(-1)^n}{F_n}, \quad S_1 = \sum_{n=1}^{\infty} \frac{1}{F_n F_{n+1}}, \quad S_1^* = \sum_{n=1}^{\infty} \frac{(-1)^n}{F_n F_{n+1}}.$$

Brousseau [7] and Rabinowitz [32] proved that

$$T_{2k} = \frac{1}{F_{2k}} \sum_{n=1}^{k} \frac{1}{F_{2n-1}}, \quad T_{2k+1} = \frac{1}{F_{2k+1}} \left(S_1 - \sum_{n=1}^{k} \frac{1}{F_{2n} F_{2n+1}} \right), \tag{2.4}$$

$$T_k^* = \frac{1}{F_k} \left(k S_1^* + \sum_{n=1}^{k} \frac{F_{n-1}}{F_n} \right). \tag{2.5}$$

Shortly after the result of Brousseau, Carlitz [11] wrote S_k and S_k^* $(k \geq 1)$ as in the following : Letting $(F)_n = F_1 F_2 \cdots F_n$, $(F)_0 = 1$, and $\left\{ \begin{matrix} k \\ j \end{matrix} \right\} = \dfrac{(F)_k}{(F)_j (F)_{k-j}} \in \mathbb{Z}$,

$$S_{4k} = S_0 \frac{(-1)^k}{(F)_{4k}} \prod_{j=1}^{2k} L_{2j-1} - \frac{1}{(F)_{4k}} \sum_{j=1}^{4k} (-1)^{\frac{1}{2}j(j-\varepsilon)} \left\{ \begin{matrix} 4k \\ j \end{matrix} \right\} \sum_{n=1}^{j} \frac{\varepsilon^n}{F_n}, \tag{2.6}$$

$$S_{4k+1} = \frac{1}{(F)_{4k}} \sum_{j=0}^{4k} (-1)^{\frac{1}{2}j(j-\varepsilon)} \left\{ \begin{matrix} 4k \\ j \end{matrix} \right\} \left(T_{4k+1-j} - \sum_{n=1}^{j} \frac{\varepsilon^n}{F_n F_{n+4k+1-j}} \right), \tag{2.7}$$

$$S_{4k+2} = S_0^* \frac{(-1)^k}{(F)_{4k+2}} \prod_{j=1}^{2k+1} L_{2j-1} - \frac{1}{(F)_{4k+2}} \sum_{j=1}^{4k+2} (-1)^{\frac{1}{2}j(j-\varepsilon)} \left\{ \begin{matrix} 4k+2 \\ j \end{matrix} \right\} \sum_{n=1}^{j} \frac{(-\varepsilon)^n}{F_n}, \tag{2.8}$$

$$S_{4k+3} = \frac{1}{(F)_{4k+2}} \sum_{j=0}^{4k+2} (-1)^{\frac{1}{2}j(j-\varepsilon)} \left\{ \begin{matrix} 4k+2 \\ j \end{matrix} \right\} \left(T_{4k+3-j}^* - \sum_{n=1}^{j} \frac{(-\varepsilon)^n}{F_n F_{n+4k+3-j}} \right), \tag{2.9}$$

where $\varepsilon = 1$ and $[x]$ is the greatest integer not exceeding x. Futhermore, $S_{4k}^*, S_{4k+1}^*, S_{4k+2}^*$, and S_{4k+3}^* are given by (2.6), (2.7), (2.8) and (2.9) with S_0^* in place of S_0, T_{4k+1-j}^* in place of T_{4k+1-j}, $-S_0$ in place of S_0^*, and T_{4k+3-j} in place of T_{4k+3-j}^*, respectively, and $\varepsilon = -1$ in any case. These formulas imply that S_k, S_k^*, T_k, and T_k^* $(k \geq 2)$ can be written as linear combinations over $\mathbb{Q}$ of 1 and one of S_0, S_1, S_0^*, and $S_1^* = \Psi$.

Bruckman and Good [9] generalized the formula (1.1) by proving that, if $r \geq 1$ is an integer,

$$\sum_{n=1}^{\infty} \frac{(-1)^{rn}}{F_{rn}F_{r(n+1)}} = \frac{\Psi^r}{{F_r}^2}. \tag{2.10}$$

The Sums S_0 and S_1 have interesting expresions. Gould [19] proved that, for any integer $r \geq 1$,

$$\sum_{n=1}^{\infty} \frac{1}{F_n} = \sum_{n=0}^{r-1} \left(\frac{1+F_{2n}}{F_{2n+1}} + \frac{2}{F_{4n+2}} - \frac{1}{\Phi} \right) + \sum_{n=0}^{\infty} \frac{1}{F_{2^r n}}. \tag{2.11}$$

The last series converges very rapidly, if r is larger. André-Jeannin [3] used the following formula to epress S_1 in terms of Lambert series (see (3.12) in section 3.2):

$$\sum_{n=1}^{\infty} \frac{1}{F_n F_{n+1}} = 2 \sum_{n=1}^{\infty} \frac{(1/\Phi)^n}{F_n} - \frac{1}{\Phi} \tag{2.12}$$

2.2 Sums in the denominators

In [5], Backstrom proved that, for any integer $r \geq 1$,

$$\sum_{n=0}^{\infty} \frac{1}{F_{2n+1} + F_{2r-1}} = \frac{(2r-1)\sqrt{5}}{2L_{2r-1}}, \tag{2.13}$$

$$\sum_{n=0}^{\infty} \frac{1}{L_{2n} + L_{2r}} = \frac{r}{\sqrt{5}F_{2r}} + \begin{cases} \dfrac{1}{2{L_r}^2} & (r \text{ even}) \\[2ex] \dfrac{1}{10{F_r}^2} & (r \text{ odd}) \end{cases}. \tag{2.14}$$

For example, if $r = 1$,

$$\sum_{n=0}^{\infty} \frac{1}{F_{2n+1} + 1} = \frac{\sqrt{5}}{2}, \quad \sum_{n=0}^{\infty} \frac{1}{L_{2n} + 3} = \frac{2\sqrt{5}+1}{10}. \tag{2.15}$$

Backstrom's proofs of these results rely heavily on the properties of Fibonacci and Lucas numbers. Popov gave in [31] some generalizations of these formulas with simpler proofs. Almkvist [1] remarked that the series used by Backstrom are in fact telescoping series. For the proof of (2.13) for example, he observes that

$$\frac{1}{F_{2n+1} + F_{2r-1}} = \frac{\sqrt{5}}{L_{2r-1}} \left(\frac{1}{1+q^{n+r+1}} - \frac{1}{1+q^{n-r}} \right), \quad q = \frac{1}{\Phi^2}.$$

Similar results were obtained by André-Jeannin in [4]. If $r \geq 1$ is an integer

$$\sum_{n=0}^{\infty} \frac{1}{F_{2n+1} + \varepsilon L_{2r}/\sqrt{5}} = \frac{\varepsilon r}{F_{2r}}, \tag{2.16}$$

$$\sum_{n=0}^{\infty} \frac{1}{L_{2n} + \varepsilon\sqrt{5}F_{2r-1}} = \frac{\varepsilon}{L_{2r-1}} \left(r - 1 + \frac{1}{1 + \varepsilon\Phi^{-2r+1}} \right), \tag{2.17}$$

where $\varepsilon = 1$. For example, if $r = 1$,

$$\sum_{n=0}^{\infty} \frac{1}{F_{2n+1} + 3/\sqrt{5}} = 1, \quad \sum_{n=0}^{\infty} \frac{1}{L_{2n} + \sqrt{5}} = \frac{1}{\Phi}.$$

Recently Zhao [33] proved that the formulas (2.16) and (2.17) hold also for $\varepsilon = -1$. These identities result from the use of telescoping series.

3 Expression by means of classical functions when the subscripts are in arithmetic progression

3.1 Use of theta functions

In [5], Backstrom gave the estimate

$$\sum_{n=0}^{\infty} \frac{1}{L_{2n} + 2} \approx \frac{1}{8} + \frac{1}{4 \log \Phi} \tag{3.1}$$

and raised the problem to compute this sum. Almkvist [1] proved that

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{1}{L_{2n} + 2} &= \frac{1}{4} + \sum_{n=1}^{\infty} \frac{q^{2n}}{(1 + q^{2n})^2}, \quad q = -\frac{1}{\Phi} \\ &= \frac{1}{8} + \frac{1}{4 \log \Phi} \left(1 - \frac{4\pi^2}{\log \Phi} \cdot \frac{\sum_{n=1}^{\infty} (-1)^n n^2 e^{-\pi^2 n^2/\log \Phi}}{1 + 2\sum_{n=1}^{\infty} (-1)^n e^{-\pi^2 n^2/\log \Phi}} \right). \end{aligned} \tag{3.2}$$

This series converges extremely rapidly. If we introduce the Jacobi's theta fucntions

$$\begin{cases} \theta_3(q) &= 1 + 2\sum_{n=1}^{\infty} q^{n^2}, \\ \theta(q) &= 1 + 2\sum_{n=1}^{\infty} (-1)^n q^{n^2}, \\ \theta_2(q) &= 2q^{1/4} \sum_{n=1}^{\infty} q^{n(n-1)}, \end{cases} \tag{3.3}$$

we see that (3.2) gives an expression of the series $\sum_{n=0}^{\infty} \frac{1}{L_{2n}+2}$ by means of the numbers $\theta(e^{-\pi^2/\log\Phi})$ and $\theta'(e^{-\pi^2/\log\Phi})$, where '$\prime$' denotes the derivation $q\frac{d}{dq}$. Almkvist's proof of (3.2) consists in using a classical formula for theta functions with suitably specialized variables. By the same method André–Jeannin [4] proved the expression

$$\sum_{n=0}^{\infty} \frac{1}{F_{2n+1}+2/\sqrt{5}} = \frac{\sqrt{5}}{4\log\Phi} - \frac{\pi^2\sqrt{5}}{(\log\Phi)^2} \frac{\theta_3'(e^{-\pi^2/\log\Phi})}{\theta_3(e^{-\pi^2/\log\Phi})}.$$

Zhao[33] obtained similar results for the series $\sum_{n=0}^{\infty} \frac{1}{F_{2n+1}-2/\sqrt{5}}$ and also for $\sum_{n=0}^{\infty} \frac{1}{L_{2n}-2}$.

The odd and even parts of $\sum_{n=1}^{\infty} \frac{1}{F_n}$ and $\sum_{n=1}^{\infty} \frac{1}{L_n}$, respectively, can be expressed in terms of Jacobi's theta functions;

$$\sum_{n=1}^{\infty} \frac{1}{F_{2n-1}} = \frac{\sqrt{5}}{2}{\theta_2}^2(\frac{1}{\Phi^2}), \tag{3.4}$$

$$\sum_{n=1}^{\infty} \frac{1}{L_{2n}} = \frac{1}{4}\left[{\theta_3}^2(\frac{1}{\Phi^2}) - 1\right]. \tag{3.5}$$

The first formula was given by Landau [26]. Catalan [12] wrote the second series in terms of the complete elliptic integral of the first kind, which is equivalent to (3.5)

In 1977, Bruckman [8] obtained closed form expressions for certain series involving hyperbolic secants and cosecants in terms of complete elliptic integrals of the first and second kind. Specializing the modulus, an implicit parameter of the integrals, and using the relations

$$\begin{cases} \frac{1}{F_{2n}} = \frac{\sqrt{5}}{2} \text{ cosech } 2n\lambda, & \frac{1}{F_{2n+1}} = \frac{\sqrt{5}}{2} \text{ sech } (2n+1)\lambda, \\ \frac{1}{L_{2n}} = \frac{1}{2} \text{ sech } 2n\lambda, & \frac{1}{L_{2n+1}} = \frac{1}{2} \text{ cosech } (2n+1)\lambda, \end{cases}$$

with $\lambda = \log\Phi$, he obtained closed form expressions for reciprocal sums of Fibonacci and Lucas numbers including those in (3.6) with $s = 1$. He remarked that the series

$$\sum_{n=1}^{\infty} \text{cosech } nx$$

cannot be evaluated by elliptic functions and so the reciprocal sums

$$\sum_{n=1}^{\infty} \frac{1}{F_{2n}} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{L_{2n-1}}.$$

Shortly later Zucker [34] proved that for any integers $s \geq 1$ the reciprocal sums

$$\sum_{n=1}^{\infty} \frac{1}{F_n{}^{2s}}, \quad \sum_{n=1}^{\infty} \frac{1}{F_{2n-1}{}^{s}}, \quad \sum_{n=1}^{\infty} \frac{1}{L_n{}^{2s}}, \quad \sum_{n=1}^{\infty} \frac{1}{L_{2n}{}^{s}}, \tag{3.6}$$

among others, can be expressed as rational functions of Jacobi's theta functions with rational coefficients. In the above-mentioned paper [1], Almkvist collected some formulas connecting, as in (3.4) and (3.5), reciprocal sums of Fibonacci and Lucas numbers and Jacobi's theta functions. The proofs in [8], [34], and [1] are based on classical identities in elliptic function theory mostly due to Jacobi.

The catalogue of formulas given by Almkvist in [1] suggested us the possibility of proving the transcendence of such sums by using Nesterenko's theorem on modular functions [17]. In the last paper we gave elementary proofs of some formulas by using Jacobi's triple product identity, for example

$$\sum_{n=1}^{\infty} \frac{n}{F_{2n}} = \frac{1}{2}\left(\frac{1}{\Phi^2} - \Phi^2\right)\frac{\theta'(1/\Phi^2)}{\theta(1/\Phi^2)}. \tag{3.7}$$

We remark that

$$\sum_{n=1}^{\infty} \frac{n}{F_{2n}} = \sqrt{5}\sum_{n=1}^{\infty} \frac{1}{L_{2n-1}{}^{2}}, \quad \sum_{n=1}^{\infty} (-1)^{n+1}\frac{n}{F_{2n}} = \frac{1}{\sqrt{5}}\sum_{n=1}^{\infty} \frac{1}{F_{2n-1}{}^{2}},$$

as proved by Jennings in [24].

3.2 Expressions by means of Lambert series

Lambert series are series of the form

$$f(x) = \sum_{n=1}^{\infty} \frac{a_n x^n}{1 - x^n}. \tag{3.8}$$

(See [15], [20], for intance.) The simplest Lambert series is

$$L(x) = \sum_{n=1}^{\infty} \frac{x^n}{1 - x^n}. \tag{3.9}$$

It has been known for a long time that

$$\sum_{n=1}^{\infty} \frac{1}{F_{2n}} = \sqrt{5}\left[L\left(\frac{1}{\Phi^2}\right) - L\left(\frac{1}{\Phi^4}\right)\right], \tag{3.10}$$

$$\sum_{n=1}^{\infty} \frac{1}{L_{2n-1}} = -L\left(-\frac{1}{\Phi}\right) + 2L\left(\frac{1}{\Phi^2}\right) - L\left(\frac{1}{\Phi^4}\right). \tag{3.11}$$

$L(-\frac{1}{\Phi}) = \sum_{n=1}^{\infty}$

André–Jeannin [3] proved that

$$\sum_{n=1}^{\infty} \frac{1}{F_n F_{n+1}} = 2\sqrt{5}\left[L(\frac{1}{\Phi^2}) - 2L(\frac{1}{\Phi^4}) + 2L(\frac{1}{\Phi^8})\right] - \frac{1}{\Phi}, \tag{3.12}$$

$$\sum_{n=1}^{\infty} \frac{1}{L_n L_{n+1}} = \frac{2}{\sqrt{5}}\left[L(\frac{1}{\Phi^2}) - 2L(\frac{1}{\Phi^8})\right] - \frac{1}{\sqrt{5}\Phi}. \tag{3.13}$$

Some generalization of these formulas can be found in Melham and Shannon [29]. A recent paper of Melham [28] gives expressions of some series involving $L(x)$. For example

$$\sum_{n=1}^{\infty} \frac{F_{2mn}}{L_{2mn} L_{4mn}} = \frac{1}{\sqrt{5}}\left[L(\frac{1}{\Phi^{4m}}) - 3L(\frac{1}{\Phi^{8m}}) + 2L(\frac{1}{\Phi^{16m}})\right]. \tag{3.14}$$

Another Lambert series is

$$H(x) = \sum_{n=1}^{\infty} \frac{nx^n}{1-x^n}. \tag{3.15}$$

This one is connected with the theory of elliptic functions and is used in the proof of some formulas. For example, we have

$$\begin{aligned} H(x) &= \sum_{n=1}^{\infty} nx^n \sum_{k=0}^{\infty} x^{nk} = \sum_{k=0}^{\infty}\sum_{n=1}^{\infty} nx^{n(k+1)} \\ &= \sum_{k=1}^{\infty} x^k \sum_{n=1}^{\infty} nx^{k(n-1)} = \sum_{k=1}^{\infty} \frac{x^k}{(1-x^k)^2} \end{aligned}$$

Therefore, with $x = -\frac{1}{\Phi^2}$

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{F_k^{\,2}} = H(-\frac{1}{\Phi^2}) \cdot \frac{\Phi^2}{(\Phi^2+1)^2}. \tag{3.16}$$

This formula leads to an elementary proof of the irrationality of the series in the left-hand side of (3.16) in [14] and, by connecting it to modular functions, to a proof of its transcendence in [17]

In [23] Horadam gave an interesting historical survey on the subjects of this section and the preceeding one.

3.3 Use of the q-exponential function

Let $q \in \mathbb{C}$ with $|q| > 1$. The q-exponential function is the most simple q-hypergeometric function (cf. [18]). It is defined by

$$E_q(x) = 1 + \sum_{n=1}^{\infty} \frac{x^n}{(q-1)\cdots(q^n-1)} \quad (x \in \mathbb{C}) \tag{3.17}$$

It is a q-analogue of the ordinary exponential function in the sense that

$$\lim_{q \to 1} E_q((q-1)x) = e^x.$$

As a special case of the q-binomial theorem [18], we have

$$E_q(x) = \prod_{n=1}^{\infty} \left(1 - \frac{x}{q^n}\right) \quad (x \in \mathbb{C}). \tag{3.18}$$

An elementary proof of (3.18) can be obtained by noting that the q-exponential function satisfies the functional equation

$$E_q(x) - E_q\left(\frac{x}{q}\right) = \frac{x}{q} E_q\left(\frac{x}{q}\right). \tag{3.19}$$

If we take the logarithmic derivative of (3.18), we get

$$\frac{E_q'(-x)}{E_q(-x)} = \sum_{k=1}^{\infty} \frac{1}{q^k - x} \tag{3.20}$$

But it is clear that

$$\sum_{n=1}^{\infty} \frac{x^n}{q^n - 1} = \sum_{n=1}^{\infty} \frac{x}{q^n - x} \quad (|x| < |q|) \tag{3.21}$$

Denote by L_q the q-logarithmic function defined by

$$\mathrm{Log}_q(x) = \sum_{n=1}^{\infty} \frac{x^n}{q^n - 1} \quad (|x| < |q|), \tag{3.22}$$

which is a q-analogue of the ordinary logarithmic function, because

$$\lim_{q \to 1} (q-1)\mathrm{Log}_q(x) = -\log(1-x).$$

By using (3.20) and (3.21), we see that

$$\mathrm{Log}_q(x) = x \frac{E_q'(-x)}{E_q(-x)} \quad (|x| < |q|). \tag{3.23}$$

This formula enables us to express the sum of the reciprocals of Fibonacci numbers in terms of a q-exponential function. Indeed, if we take $q = -\Phi^2$, we have

$$\sum_{n=1}^{\infty} \frac{z^n}{F_n} = (\Phi - \Psi) \sum_{n=1}^{\infty} \frac{(-\Phi z)^n}{q^n - 1} \tag{3.24}$$

$$= (\Phi - \Psi)\Phi z \frac{E_q{}'(\Phi z)}{E_q(\Phi z)} \quad (|z| < \Phi). \tag{3.25}$$

In particular, the numbers S_0, S_0^*, and S_1 defined in section 2.1 can be written as

$$\sum_{n=1}^{\infty} \frac{1}{F_n} = (\Phi - \Psi) \sum_{n=1}^{\infty} \frac{(-\Phi)^n}{q^n - 1} = (\Phi - \Psi)\Phi \frac{E_q{}'(\Phi)}{E_q(\Phi)}, \tag{3.26}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{F_n} = (\Phi - \Psi) \sum_{n=1}^{\infty} \frac{\Phi^n}{q^n - 1} = -(\Phi - \Psi)\Phi \frac{E_q{}'(-\Phi)}{E_q(-\Phi)}, \tag{3.27}$$

and

$$\sum_{n=1}^{\infty} \frac{1}{F_n F_{n+1}} = 2(\Phi - \Psi) \sum_{n=1}^{\infty} \frac{(-1)^n}{q^n - 1} - \frac{1}{\Phi} = 2(\Phi - \Psi) \frac{E_q(1)'}{E_q(1)} - \frac{1}{\Phi}, \tag{3.28}$$

by using (2.12) or by direct calculation. The formula (3.26) seems to appear first in [10]. The first named author of the present paper used it in [13] to prove the irrationality of the sum $\sum_{n=1}^{\infty} \frac{1}{F_n}$.

4 Computation in closed form when the subscripts are in geometric progression

4.1 Some known results

The formula (1.2) of Lucas has been rediscovered in many papers. For example, Hoggatt and Bicknel [21] gave eleven different methods for finding the value of the sum (1.2). Shortly later, they proved in [22] a more general formula

$$\sum_{n=0}^{\infty} \frac{1}{F_{k2^n}} = \frac{1}{F_k} + \frac{\Phi^2 + 1}{\Phi(\Phi^{2k} - 1)}, \tag{4.1}$$

which is also an easy consequence of formula (1.3).

Bruckman and Good [9] obtained symmetric formulas

$$\sum_{n=0}^{\infty} \frac{L_{k3^n}}{F_{k3^{n+1}}} = \frac{1}{\Phi^k F_k}, \quad \sum_{n=0}^{\infty} \frac{F_{k3^n}}{L_{k3^{n+1}}} = \frac{1}{\sqrt{5}\Phi^k L_k},$$

using a generalized de Morgan's series for an integer $d \geq 2$

$$\sum_{n=0}^{\infty} \frac{x^{d^n}(1 - x^{(d-1)d^n})}{(1 - x^{d^n})(1 - x^{d^{n+1}})} = \frac{x}{1-x} \tag{4.2}$$

with $d = 3$ and $x = (\Psi/\Phi)^k$.

Other results of a similar kind are

$$\sum_{n=1}^{\infty} \frac{4^n}{L_{2^n} + 2} = 4, \tag{4.3}$$

$$\sum_{n=1}^{\infty} \frac{(-2)^n}{L_{2^n} - 1} = \frac{-2}{L_2 + 1} = -\frac{1}{2} \tag{4.4}$$

(see for example [16] and [6], respectively). It is interesting to remark that (1.2), (4.3), (4.4) result, respectively, from sums of series of rational functions

$$\sum_{n=0}^{\infty} \frac{x^{2^n}}{1 - x^{2^{n+1}}} = \frac{x}{1-x}, \tag{4.5}$$

$$\sum_{n=0}^{\infty} \frac{4^n x^{2^n}}{(1 + x^{2^n})^2} = \frac{x}{(1-x)^2}, \tag{4.6}$$

$$\sum_{n=0}^{\infty} \frac{(-2)^n x^{2^n}}{x^{2^{n+1}} - x^{2^n} + 1} = \frac{x}{x^2 + x + 1}. \tag{4.7}$$

with $x = \Psi$. We note that Jennings [25] proved (4.6) and deduced

$$\sum_{n=1}^{\infty} \left(\frac{2^n}{L_{2^n}}\right)^2 = \frac{4}{5},$$

which is equivalent to (4.3), since ${L_{2^n}}^2 = L_{2^{n+1}} + 2$. Another identity of the same type is

$$\sum_{n=0}^{\infty} \frac{2^n x^{2^n}}{1 + x^{2^n}} = \frac{x}{1-x}, \tag{4.8}$$

which however gives a less interesting formula when applied to $x = \Psi$. These identities are listed and proved in [16], Theorem 2.7.

4.2 A general identity

Our two questions are: Are these identities (4.2), (4.5)–(4.8) connected? Is there a way for obtaining them all together?

On what concerns the first question, it is clear that (4.6) comes from (4.8) by term-by-term differentiation. Also, (4.7) can be obtained from (4.8); replace first x by ρx with $\rho = e^{2\pi i/3}$ then replace x by $\bar{\rho} x$, and add. Clearly, (4.2) with $d = 2$ yields (4.5). But at first sight, it seems difficults to connect (4.5) and (4.8). As for the second questionm the following formula (4.11) enables us to obtain (4.5), and (4.8) together and to give new Fibonacci identities. Furthermore, the formula (4.2) and also (4.11) can be deduced form a general identity (4.12).

Theorem 4.1. *Let $c, d \in \mathbb{Z}, d \geq 2, c \neq 0$. Let $P, Q \in \mathbb{C}[x]$ satisfying $P(0) = Q(0) = 1$ and*

$$P(x^d) = P(x)Q(x). \tag{4.9}$$

Then for $|x| < 1$

$$\prod_{n=0}^{\infty} \left[\frac{Q(x^{d^n})}{(P(x^{d^n}))^{c-1}} \right]^{\frac{1}{c^n}} = \frac{1}{P(x)^c}. \tag{4.10}$$

Proof. If c=1, the left–hand side is

$$\prod_{n=0}^{\infty} Q(x^{d^n}) = \prod_{n=0}^{\infty} \frac{P(x^{d^{n+1}})}{P(x^{d^n})} = \frac{1}{P(x)}$$

by (4.9). Let $c \neq 1$. Then we have using (4.9)

$$\prod_{n=0}^{\infty} P(x^{d^n})^{\frac{1}{c^n}} = \prod_{n=0}^{\infty} \left[P(x) \prod_{k=0}^{n-1} Q(x^{d^k}) \right]^{\frac{1}{c^n}} = P(x)^{\frac{c}{c-1}} \prod_{n=1}^{\infty} \prod_{k=0}^{n-1} Q(x^{d^k})^{\frac{1}{c^n}}$$

with

$$\prod_{n=1}^{\infty} \prod_{k=0}^{n-1} Q(x^{d^k})^{\frac{1}{c^n}} = \prod_{k=0}^{\infty} \prod_{n=k+1}^{\infty} Q(x^{d^k})^{\frac{1}{c^n}} = \prod_{k=0}^{\infty} Q(x^{d^k})^{\frac{1}{c^k(c-1)}}.$$

Hence we get

$$\prod_{n=0}^{\infty} P(x^{d^n})^{\frac{1}{c^n}} = P(x)^{\frac{c}{c-1}} \prod_{n=0}^{\infty} Q(x^{d^n})^{\frac{1}{c^n(c-1)}}$$

and (4.10) follows.

If we take the logarithmic derivative of (4.10) and multiply by x we immediately obtain

Corollary 4.1. *Let c, d, P, Q be as in Theorem 4.1. Then for every $|x| < 1$*

$$\sum_{n=0}^{\infty}\left(\frac{d}{c}\right)^{n} x^{d^n}\frac{Q'(x^{d^n})P(x^{d^n})-(c-1)P'(x^{d^n})Q(x^{d^n})}{P(x^{d^{n+1}})} = -cx\frac{P'(x)}{P(x)}. \tag{4.11}$$

This formula appears as a generalization of old result of Jacobi (see for example [30], 164 p.30). In an earlier version of this paper, Taka–aki Tanaka remarked that, if $a \in \mathbb{C}^{\times}$ and $G(x), H(x) \in \mathbb{C}[x]$ with $G(0) = 0$ and $H(0) = 1$,

$$\sum_{n=0}^{\infty}\left(a^n\frac{G(x^{d^n})}{H(x^{d^n})} - a^{n+1}\frac{G(x^{d^{n+1}})}{H(x^{d^{n+1}})}\right) = \frac{G(x)}{H(x)}, \tag{4.12}$$

and (4.11) can be obtained by putting $a = d/c, G(x) = xP'(x), H(x) = P(x)$.

4.3 Examples

Example 4.1. Take $d = c = 2, P(x) = 1 - x$, and $Q(x) = 1 + x$. Replacing in (4.11) yields (4.2).

Example 4.2. Take $a_n = 1, b_n = d^n, f(x) = x$, and $g(x) = 1 - x$. Replacing in (4.12) yields (4.2).

Example 4.3. Take $d = 2, a = 1, P(x) = 1 - x$, and $Q(x) = 1 + x$. Replacing in (4.11) yields (4.8)

Example 4.4. Take $d = 3, c = 1, P(x) = 1 - x$, and $Q(x) = x^2 + x + 1$. Replacing in (4.11) yields

$$\sum_{n=0}^{\infty}\frac{3^n x^{3^n}(2x^{3^n}+1)}{x^{2\cdot 3^n}+x^{3^n}+1} = \frac{x}{1-x}. \tag{4.13}$$

We can obtain more symmetric expressions, if we replace x by $-x$ in (4.13)

$$\sum_{n=0}^{\infty}\frac{3^n x^{3^n}(2x^{3^n}-1)}{x^{2\cdot 3^n}-x^{3^n}+1} = -\frac{x}{1+x} \tag{4.14}$$

and substract (4.14) from (4.13); we get

$$\sum_{n=0}^{\infty}\frac{3^n x^{3^n}(1-x^{2\cdot 3^n})}{x^{4\cdot 3^n}+x^{2\cdot 3^n}+1} = \frac{x}{1-x^2}. \tag{4.15}$$

We do not know if this is a known series. From it we can get another series. Observe that $i^{3^n} = (-1)^n i$ and replace x by ix in (4.15); we obtain

$$\sum_{n=0}^{\infty}\frac{(-3)^n x^{3^n}(1+x^{2\cdot 3^n})}{x^{4\cdot 3^n}-x^{2\cdot 3^n}+1} = \frac{x}{1+x^2}. \tag{4.16}$$

As a numerical application, we take $x = 1/\Phi$ in (4.15) and (4.16). Then we get

$$\sum_{n=0}^{\infty} \frac{3^n L_{3^n}}{L_{2\cdot 3^n} + 1} = 1, \tag{4.17}$$

$$\sum_{n=0}^{\infty} \frac{(-3)^n F_{3^n}}{L_{2\cdot 3^n} - 1} = \frac{1}{5}. \tag{4.18}$$

Example 4.5. Take $d = c = 2, P(x) = 1+x+x^2$, and $Q(x) = 1-x+x^2$. Replacing in (4.11) yields

$$\sum_{n=0}^{\infty} x^{2^n} \frac{x^{2^{n+1}} - 1}{x^{2^{n+2}} + x^{2^{n+1}} + 1} = -\frac{x(2x+1)}{x^2+x+1}.$$

Putting $x = \frac{1}{\Phi}$, we have

$$\sum_{n=0}^{\infty} \frac{\Phi^{2^n} - \left(\frac{1}{\Phi}\right)^{2^n}}{1 + \Phi^{2^{n+1}} + \left(\frac{1}{\Phi}\right)^{2^{n+1}}} = \frac{1}{\Phi} \cdot \frac{\Phi+2}{1+\Phi+\frac{1}{\Phi}},$$

and so

$$\frac{\Phi - \frac{1}{\Phi}}{1+\Phi^2+\frac{1}{\Phi^2}} + \sum_{n=1}^{\infty} \frac{\sqrt{5}F_{2^n}}{1+L_{2^{n+1}}} = \frac{1}{\Phi} \cdot \frac{\Phi+2}{1+\Phi+\frac{1}{\Phi}}.$$

We finally obtain

$$\sum_{n=1}^{\infty} \frac{F_{2^n}}{L_{2^{n+1}} + 1} = \frac{1}{4\sqrt{5}}. \tag{4.19}$$

Example 4.6. Take $d = 3, c = 2, P(x) = 1 - x$, and $Q(x) = 1 + x + x^2$. Replacing in (4.11) yields

$$\sum_{n=0}^{\infty} \left(\frac{3}{2}\right)^n x^{3^n} \frac{2 + 2x^{3^n} - x^{2\cdot 3^n}}{1 - x^{3^{n+1}}} = \frac{2x}{1-x}.$$

As in Example 4.4, this can be made more symmetric by replacing x by $-x$ and substracting; we obtain

$$\sum_{n=0}^{\infty} \left(\frac{3}{2}\right)^n x^{3^n} \frac{2 - x^{2\cdot 3^n} + 2x^{4\cdot 3^n}}{1 - x^{2\cdot 3^{n+1}}} = \frac{2x}{1-x^2}. \tag{4.20}$$

Replacing x by ix, we also get

$$\sum_{n=0}^{\infty} \left(-\frac{3}{2}\right)^n x^{3^n} \frac{2 + x^{2\cdot 3^n} + 2x^{4\cdot 3^n}}{1 + x^{2\cdot 3^{n+1}}} = \frac{2x}{1+x^2}. \tag{4.21}$$

Finally, if we replace x by $\frac{1}{\Phi}$ in (4.20) and (4.21), we obtain two beautiful series involving Fibonacci and Lucas sequences

$$\sum_{n=1}^{\infty}\left(\frac{3}{2}\right)^n\frac{2L_{2\cdot 3^n}-1}{L_{3^{n+1}}}=1, \tag{4.22}$$

$$\sum_{n=1}^{\infty}\left(-\frac{3}{2}\right)^n\frac{2L_{2\cdot 3^n}+1}{F_{3^{n+1}}}=2. \tag{4.23}$$

As far as we know, the series (4.17), (4.18), (4.19), (4.22), and (4.23) are new. It is clear that many series involving Fibonacci and Lucas numbers could be computed from (4.11) and also from (4.12), but we do not want to take up too much of the reader's time!

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