# Refinement of the Chowla-Erdős method and linear independence of certain Lambert series 

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#### Abstract

In this paper, we refine the method of Chowla and Erdős on the irrationality of Lambert series and study a necessary condition for the infinite series $\sum \theta(n) / q^{n}$ to be a rational number, where $q$ is an integer with $|q|>1$ and $\theta$ is an arithmetic function with suitable divisibility and growth conditions. As applications of our main theorem, we give linear independence results for various kinds of Lambert series.


Keywords: Irrationality, Linear independence, Lambert series.
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## 1 Introduction and main results

The story of what we call "Chowla-Erdős method" in the title of this paper begins with a result of Chowla [2] dating back to 1947. In [2], Chowla proved that the number

$$
\begin{equation*}
\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{t^{2 n-1}-1}=\frac{1}{4} \sum_{n=1}^{\infty} \frac{r(n)}{t^{n}} \tag{1.1}
\end{equation*}
$$

is irrational for any integer $t \geq 5$, where $r(n)$ is the number of representations of $n$ as a sum of two squares. He showed that the base- $t$ representation of the infinite series in the right-hand side of (1.1) contains arbitrarily long strings of 0 's without being identically zero from some point on, and is therefore not ultimately periodic. Chowla also conjectures that for any rational number $t$ with $|t|>1$ the numbers (1.1) and

$$
\begin{equation*}
\sum_{n=1}^{\infty} \frac{1}{t^{n}-1}=\sum_{n=1}^{\infty} \frac{d(n)}{t^{n}} \tag{1.2}
\end{equation*}
$$

would be irrational, where $d(n)$ is the classical divisor function defined by

$$
\begin{equation*}
d(n):=\sum_{d \mid n} 1 \quad n \geq 1 . \tag{1.3}
\end{equation*}
$$

In 1948, Erdős [7] extended Chowla's result by showing that both numbers (1.1) and (1.2) are irrational for any integer $t>1$. Erdős's proof consists in using divisibility properties of the arithmetic functions $r(n)$ and

[^0]$d(n)$, and recently his method has been applied to various kinds of Lambert series; see [6], [10], [11], [12], [14]. Erdős himself published in 1969 a second paper [8] on the irrationality of Lambert series by using similar but slightly different ideas. A striking result in [8] is to establish the irrationality of the number
$$
\sum_{p: \text { prime }}^{\infty} \frac{1}{t^{p^{2}}-1}
$$
where $t \geq 2$ is an integer and the sum is taken over all prime numbers.
Remark 1.1. The above Chowla's conjecture is true for the number (1.1). Even better, it is known that this number is transcendental for every algebraic number $t$ with $|t|>1$. This follows from the well-known identity
$$
\vartheta(x)^{2}=1+\sum_{n=1}^{\infty} r(n) x^{n}, \quad|x|<1
$$
where $\vartheta(x)$ is the theta function defined by $\vartheta(x):=1+2 \sum_{n=1}^{\infty} x^{n^{2}}$, and the fact that the value $\vartheta(\alpha)$ is transcendental for any algebraic number $\alpha$ with $0<|\alpha|<1$ (cf. [13, Corollary 4.7], see also [1], [5]). Besides, an elementary proof of the irrationality of the number (1.1) for any integer $t(|t|>1)$ has been given in [3]. In contrast, the conjecture for the number (1.2) is still open. One only knows by [4, Theorem 2] that the number (1.2) is irrational for any nonzero rational number $t=r / s(r, s \in \mathbb{Z})$ satisfying
$$
\frac{\log |s|}{\log |r|}<\frac{1}{3}\left(1-\frac{3}{\pi^{2}}\right) .
$$

Note that we are still unaware of transcendence for the Erdös-Borwein constant $\sum_{n=1}^{\infty} 1 /\left(2^{n}-1\right)$.
The purpose of this paper is to refine the method of Chowla-Erdős and give linear independence results for certain series. Let $\mathcal{E}$ denote the set of all increasing sequences $E=\left\{e_{1}, e_{2}, \ldots, e_{n}, \ldots\right\}$ of positive integers greater than one which satisfy the following two conditions;

$$
\operatorname{gcd}\left(e_{i}, e_{j}\right)=1 \quad(i \neq j)
$$

and there exists a constant $\mu>1$ such that

$$
\begin{equation*}
e_{n} \leq n^{\mu} \tag{1.4}
\end{equation*}
$$

holds for any large $n$. For example, the sequence of all prime numbers belongs to the set $\mathcal{E}$, since the $n$th prime number $p_{n}$ is asymptotic to $n \log n$. Throughout this paper, let $q$ be an integer with $|q|>1$.

Theorem 1.1. Assume that the arithmetic function $\theta: \mathbb{Z}_{>0} \rightarrow \mathbb{Z}$ satisfies the following two conditions; $\left(H_{1}\right)$ There exists a sequence $E:=\left\{e_{n}\right\}_{n \geq 1} \in \mathcal{E}$ and a positive integer $\gamma$ such that the following property holds: if the integer $n$ has the form

$$
\begin{equation*}
n=\left(e_{i_{1}} e_{i_{2}} \cdots e_{i_{m}}\right)^{\gamma} N \quad \text { with } \quad \operatorname{gcd}\left(e_{i_{1}} e_{i_{2}} \ldots e_{i_{m}}, N\right)=1, \tag{1.5}
\end{equation*}
$$

for large distinct integers $e_{i_{1}}, e_{i_{2}}, \ldots, e_{i_{m}}$ in $E$, then $\theta(n)$ is divisible by $q^{m}$.
$\left(H_{2}\right)$ There exists a positive constant $\nu$ such that

$$
\sum_{i=0}^{n}|\theta(a i+b)| \leq n(2+\log n)^{\nu}, \quad n \geq \max \{a, b\}
$$

holds uniformly for all coprime positive integer pairs $a, b$.
Assume moreover that the infinite series

$$
\begin{equation*}
f(q):=\sum_{n=1}^{\infty} \frac{\theta(n)}{q^{n}} \tag{1.6}
\end{equation*}
$$

is rational. Then for any positive integer pair $A, B$, there exist infinitely many positive integers $n$ such that

$$
\left\{\begin{array}{l}
\theta(n)=0  \tag{1.7}\\
n \equiv B \quad(\bmod A)
\end{array}\right.
$$

As an application of Theorem 1.1, we obtain the following Theorem 1.2. In what follows, let $h$ and $\ell$ be positive integers.
Theorem 1.2. Assume that all the arithmetic functions $\theta_{1}, \theta_{2}, \ldots, \theta_{\ell}$ satisfy the two conditions $\left(H_{1}\right)$ for a fixed $E \in \mathcal{E}$ and a fixed positive integer $\gamma$, and $\left(\mathrm{H}_{2}\right)$ in Theorem 1.1. If the $h \ell+1$ numbers

$$
\begin{equation*}
1, \quad \sum_{n=1}^{\infty} \frac{\theta_{i}(n)}{q^{j n}} \quad(i=1,2, \ldots, \ell, j=1,2, \ldots, h) \tag{1.9}
\end{equation*}
$$

are linearly dependent over $\mathbb{Q}$, then there exist integers $\xi_{1}, \xi_{2}, \ldots, \xi_{\ell}$, not all zero, such that the following property holds: For any positive integer pair $A, B$ with $\operatorname{gcd}(A, h!)=1$, there exist infinitely many positive integers $n$ such that

$$
\left\{\begin{array}{l}
\xi_{1} \theta_{1}(n)+\xi_{2} \theta_{2}(n)+\cdots+\xi_{\ell} \theta_{\ell}(n)=0 \\
n \equiv B \quad(\bmod A)
\end{array}\right.
$$

Note that Theorem 1.2 with $\ell=1$ shows that the $h+1$ numbers

$$
\begin{equation*}
1, \quad \sum_{n=1}^{\infty} \frac{\theta(n)}{q^{n}}, \quad \sum_{n=1}^{\infty} \frac{\theta(n)}{q^{2 n}}, \quad \ldots, \quad \sum_{n=1}^{\infty} \frac{\theta(n)}{q^{h n}} \tag{1.10}
\end{equation*}
$$

are linearly independent over $\mathbb{Q}$ for any $\theta$ such that $\theta(n)$ does not vanish for large $n$ and satisfies the conditions $\left(H_{1}\right)$ and $\left(H_{2}\right)$.
Corollary 1.1. Let $d(n)$ be the divisor function defined by (1.3) and let $\left\{a_{n}\right\}_{n \geq 1}$ be a sequence of nonzero integers satisfying $\log \left|a_{n}\right|=O(\log \log n)$. Then for every integer $h \geq 1$ the numbers

$$
1, \quad \sum_{n=1}^{\infty} \frac{d(n) a_{n}}{q^{n}}, \quad \sum_{n=1}^{\infty} \frac{d(n) a_{n}}{q^{2 n}}, \quad \ldots, \quad \sum_{n=1}^{\infty} \frac{d(n) a_{n}}{q^{h n}}
$$

are linearly independent over $\mathbb{Q}$.
Corollary 1.1 generalizes a result of J. Vandehey [14, Theorem 1.2], who proved the irrationality of the number $\sum_{n=1}^{\infty} d(n) b_{n} / q^{n}$ for a bounded sequence of nonzero integers $\left\{b_{n}\right\}_{n \geq 1}$.

For $E:=\left\{e_{n}\right\}_{n \geq 1} \in \mathcal{E}$ and $s \in \mathbb{Z}_{\geq 2} \cup\{\infty\}$, we define

$$
\begin{equation*}
F_{s}:=F_{s}(E):=\left\{f_{1}, f_{2}, \ldots, f_{n}, \ldots\right\} \tag{1.11}
\end{equation*}
$$

as the increasing sequence of all integers of the form $f_{n}=\prod_{i} e_{i}^{m_{i}}$, where the product is taken over finitely many values $i$ and the integers $m_{i}$ with $0 \leq m_{i}<s$ (resp. $0 \leq m_{i}$, if $s:=\infty$ ). Note that $1 \in F_{s}$ for any sequence $E \in \mathcal{E}$. Let $\mathbb{P} \in \mathcal{E}$ be the sequence of all prime numbers. Then, for example, the sequences $F_{2}(\mathbb{P})$, $F_{3}(\mathbb{P}), F_{\infty}(\mathbb{P})$ consist of all squarefree, cubefree, and positive integers, respectively. Now Theorem 1.2 gives the following linear independence results for certain Lambert series.

Corollary 1.2. Let $E \in \mathcal{E}$ be given and $F_{s}$ be defined in (1.11) for a fixed $s \in \mathbb{Z}_{\geq 2} \cup\{\infty\}$. Let $q(|q|>1)$ be an integer satisfying $|q| L \leq s$, where $L:=\operatorname{lcm}(1,2, \ldots, \ell)$. Then the numbers

$$
\begin{equation*}
1, \quad \sum_{n \in F_{s}} \frac{1}{q^{j n^{i}}-1} \quad(i=1,2, \ldots, \ell, j=1,2, \ldots, h) \tag{1.12}
\end{equation*}
$$

are linearly independent over $\mathbb{Q}$. In particular, if $s:=\infty$, then the numbers (1.12) are linearly independent over $\mathbb{Q}$ for any integer $q(|q|>1)$. The same holds for the numbers

$$
1, \quad \sum_{n \in F_{s}} \frac{1}{q^{j n^{i}}+1} \quad(i=1,2, \ldots, \ell, j=1,2, \ldots, h)
$$

We give some examples of Corollary 1.2.
Example 1.1. Let $\mu(n)$ be the Möbius function. By putting $F_{2}:=F_{2}(\mathbb{P})$ and $\ell=1$, we see that the numbers

$$
1, \quad \sum_{n=1}^{\infty} \frac{|\mu(n)|}{2^{n}-1}, \quad \sum_{n=1}^{\infty} \frac{|\mu(n)|}{2^{2 n}-1}, \quad \ldots, \quad \sum_{n=1}^{\infty} \frac{|\mu(n)|}{2^{h n}-1}
$$

are linearly independent over $\mathbb{Q}$. It is intriguing to compare this result with the fact that the number $\sum_{n=1}^{\infty} \mu(n) /\left(2^{n}-1\right)=1 / 2$ is a rational number.
Example 1.2. Let $E_{1}$ and $E_{2}$ be the sequences of prime numbers congruent to 1 modulo 4 and of the squares of prime numbers congruent to 3 modulo 4, respectively. Then $E:=\{2\} \cup E_{1} \cup E_{2}$ belongs to $\mathcal{E}$ and $F_{\infty}:=F_{\infty}(E)$ consists of all positive integers $s_{n}(n \geq 1)$ which can be expressed as a sum of two squares. Then the numbers

$$
1, \quad \sum_{n=1}^{\infty} \frac{1}{q^{j s_{n}^{i}}-1} \quad(i=1,2, \ldots, \ell, j=1,2, \ldots, h)
$$

are linearly independent over $\mathbb{Q}$ for any integer $q(|q|>1)$.
Example 1.3. Let $N \geq 1$ be an integer and $E \in \mathcal{E}$ be the sequence of prime numbers coprime to $N$. Then $F_{\infty}:=F_{\infty}(E)$ consists of all positive integers coprime to $N$, and the numbers

$$
\begin{equation*}
1, \quad \sum_{\substack{n=1 \\(n, N)=1}}^{\infty} \frac{1}{q^{j n^{i}}-1} \quad(i=1,2, \ldots, \ell, j=1,2, \ldots, h) \tag{1.13}
\end{equation*}
$$

are linearly independent over $\mathbb{Q}$ for any integer $q(|q|>1)$.
The second author and F. Luca [10], [11] gave linear independence results for some subsets of the numbers (1.13) by using a result on primes in arithmetic progression with large moduli. Erdôs and Graham conjecture in [9, p. 62] that the number $\sum_{k=1}^{\infty} 1 /\left(2^{n_{k}}-1\right)$ is irrational for any increasing sequence of positive integers $\left\{n_{k}\right\}_{k \geq 1}$. Corollary 1.2 gives irrationality of the numbers $\sum_{n \in \mathcal{A}} 1 /\left(2^{n}-1\right)$ for a large variety of the sets $\mathcal{A}$, and support for their conjecture.

The structure of this paper is as follows. In Section 2, we give the proof of our main Theorem 1.1 based on elementary arguments used in the papers of Chowla [2] and Erdős [7]. Theorem 1.2 will be proved in Section 3. Section 4 is devoted to the proofs of Corollary 1.1 and 1.2.

## 2 Proof of Theorem 1.1

Let $A$ and $B$ be any positive integers and $\theta(n)$ be the arithmetic function satisfying the two conditions $\left(H_{1}\right)$ and $\left(H_{2}\right)$ in Theorem 1.1. Let $E:=\left\{e_{n}\right\}_{n \geq 1} \in \mathcal{E}$ and $\gamma$ be a positive integer given in the condition $\left(H_{1}\right)$. In what follows, let $k$ be a positive integer sufficiently large. Let $S_{k}:=\left\{e_{k^{3}+1}, e_{k^{3}+2}, \ldots, e_{k^{9}}\right\}$ and $\rho_{j}$ be as the smallest prime divisor of $e_{j}$ (recall that $e_{j} \geq 2$ by definition). Since the integers $e_{j}$ 's are pairwise coprime, the set $S_{k}$ contains at most $k^{6}-2$ numbers $e_{j}$ such that $1<\rho_{j}<k^{6}$. Hence, dividing the set $S_{k}$ into the $k^{6}-1$ subsets

$$
\left\{e_{j k^{3}+1}, e_{j k^{3}+2}, \ldots, e_{(j+1) k^{3}}\right\}, \quad j=1,2, \ldots, k^{6}-1
$$

we find that there exists an integer $j_{k}$ with $1 \leq j_{k} \leq k^{6}-1$ such that

$$
\begin{equation*}
\rho_{j}>k^{6} \tag{2.1}
\end{equation*}
$$

holds for the consecutive $k^{3}$ integers $j=j_{k} k^{3}+1, \ldots,\left(j_{k}+1\right) k^{3}$. We fix the least such integer $j_{k}$ and define

$$
\begin{equation*}
L_{i}:=L_{i}(k):=e_{j_{k} k^{3}+2 k(i-1)+1} \cdots e_{j_{k} k^{3}+2 k i}, \quad i=1,2, \ldots, 2 k \tag{2.2}
\end{equation*}
$$

Consider the system of simultaneous congruences

$$
\left\{\begin{array}{rlll}
X-m & \equiv L_{m}^{\gamma} & \left(\bmod L_{m}^{\gamma+1}\right), & m=1,2, \ldots, k,  \tag{2.3}\\
X & \equiv B & (\bmod A), & \\
X+m & \equiv L_{k+m}^{\gamma} & \left(\bmod L_{k+m}^{\gamma+1}\right), & m=1,2, \ldots, k,
\end{array}\right.
$$

where by (2.1) the integers $A, L_{1}, L_{2}, \ldots, L_{2 k}$ are pairwise coprime for large $k$. Hence, by the Chinese Remainder Theorem, there exists a unique integer solution $X:=\eta_{k}$ of (2.3) satisfying $A H_{k}<\eta_{k} \leq 2 A H_{k}$, where

$$
\begin{equation*}
H_{k}:=\prod_{i=1}^{2 k} L_{i}^{\gamma+1}=\prod_{i=1}^{4 k^{2}} e_{j_{k} k^{3}+i}^{\gamma+1} \tag{2.4}
\end{equation*}
$$

Since $j_{k} \geq 1$, we obtain from the assumption (1.4) that for some constant $\mu>1$

$$
H_{k} \leq \prod_{i=1}^{4 k^{2}}\left(j_{k} k^{3}+i\right)^{(\gamma+1) \mu} \leq \prod_{i=1}^{4 k^{2}}\left(k^{9}+4 k^{2}\right)^{(\gamma+1) \mu} \leq k^{40(\gamma+1) \mu k^{2}}
$$

so that

$$
\begin{equation*}
A H_{k}<\eta_{k} \leq 2 A H_{k} \leq 2^{k^{3}} \tag{2.5}
\end{equation*}
$$

Let $M_{k}:=2^{k^{4}}$ and

$$
\begin{equation*}
u_{k, i}:=i A H_{k}+\eta_{k}, \quad i=0,1, \ldots, M_{k} . \tag{2.6}
\end{equation*}
$$

We observe that the two integers $H_{k}$ and $\eta_{k}+m$ are coprime for each of the integers $m=0$ and $m=$ $k+1, k+2, \ldots, 2 k^{5}$. Otherwise, there exists a common prime factor $p$ of $H_{k}$ and $\eta_{k}+m$. Then $H_{k}$ is divisible by $p$ and so is one of the $e_{i}$ 's for $j_{k} k^{3}+1 \leq i \leq j_{k} k^{3}+4 k^{2}$, which implies $p>k^{6}$ by (2.1). Moreover, $\eta_{k}+m^{\prime}$ is also divisible by $p$ for some $m^{\prime}\left(1 \leq\left|m^{\prime}\right| \leq k\right)$ by the congruences (2.3). Hence, $m-m^{\prime}$ is divisible by $p$, since $p$ is a prime factor of $\eta_{k}+m$. Since $0<\left|m-m^{\prime}\right| \leq 2 k^{5}+k$, we have $p \leq 2 k^{5}+k$, which is impossible for large $k$. Let

$$
\mu_{k, i}:=\left|\theta\left(u_{k, i}\right)\right|+\sum_{m=k+1}^{2 k^{5}}\left|\theta\left(u_{k, i}+m\right)\right|, \quad i=0,1, \ldots, M_{k}
$$

and $\tau_{k}$ be the minimum of the $M_{k}$ integers $\mu_{k, 0}, \mu_{k, 1}, \ldots, \mu_{k, M_{k}}$. Then we have by the condition $\left(H_{2}\right)$

$$
\begin{aligned}
\left(M_{k}+1\right) \tau_{k} & \leq \sum_{i=0}^{M_{k}}\left|\theta\left(i A H_{k}+\eta_{k}\right)\right|+\sum_{m=k+1}^{2 k^{5}} \sum_{i=0}^{M_{k}}\left|\theta\left(i A H_{k}+\eta_{k}+m\right)\right| \\
& \leq \sum_{i=0}^{A M_{k}}\left|\theta\left(i H_{k}+\eta_{k}\right)\right|+\sum_{m=k+1}^{2 k^{5}} \sum_{i=0}^{A M_{k}}\left|\theta\left(i H_{k}+\eta_{k}+m\right)\right| \\
& \leq 2 k^{5} A M_{k}\left(2+\log A M_{k}\right)^{\nu},
\end{aligned}
$$

where $\nu$ is a positive constant given in the condition $\left(H_{2}\right)$, so that

$$
\begin{equation*}
\tau_{k} \leq k^{4 \nu+6} \tag{2.7}
\end{equation*}
$$

since $M_{k} \leq e^{k^{4}}$. Let $i_{k}$ be the least integer such that $\mu_{k, i_{k}}=\tau_{k}$ and

$$
\begin{equation*}
n_{k}:=u_{k, i_{k}}=i_{k} A H_{k}+\eta_{k} . \tag{2.8}
\end{equation*}
$$

By (2.4) we have $n_{k} \geq \eta_{k}>A H_{k}>2^{4 k^{2}}$. It follows from (2.7) and (2.8) that

$$
\begin{equation*}
\sum_{m=k+1}^{2 k^{5}}\left|\theta\left(n_{k}+m\right)\right| \leq \tau_{k} \leq k^{4 \nu+6}, \quad\left|\theta\left(n_{k}\right)\right| \leq \tau_{k} \leq k^{4 \nu+6} \tag{2.9}
\end{equation*}
$$

Now we complete the proof of Theorem 1.1 by showing that the integers $n:=n_{k}$ satisfy the properties (1.7) and (1.8) for any large $k$. The property (1.8) is clear, since $n_{k} \equiv B(\bmod A)$ holds for every $k$. We prove that $\theta\left(n_{k}\right)=0$ for any large $k$. Clearly, $X:=n_{k}$ is a solution of the system of simultaneous congruences (2.3), so that each integer $n_{k}+m(1 \leq|m| \leq k)$ has the form (1.5) for large distinct $2 k$ integers $e_{j}$ given in (2.2). Hence, by the condition $\left(H_{1}\right)$ the integers $\theta\left(n_{k}+m\right)$ are divisible by $q^{2 k}$ for all integers $m$ with $1 \leq|m| \leq k$, and the infinite series (1.6) is written as

$$
\begin{align*}
f(q) & =\sum_{n=1}^{n_{k}-k-1} \frac{\theta(n)}{q^{n}}+\sum_{n=n_{k}-k}^{n_{k}-1} \frac{\theta(n)}{q^{n}}+\frac{\theta\left(n_{k}\right)}{q^{n_{k}}}+\sum_{n=n_{k}+1}^{n_{k}+k} \frac{\theta(n)}{q^{n}}+\sum_{n=n_{k}+k+1}^{\infty} \frac{\theta(n)}{q^{n}}, \\
& =\frac{a_{k}}{q^{n_{k}-k}}+\frac{\theta\left(n_{k}\right)}{q^{n_{k}}}+V_{k}, \tag{2.10}
\end{align*}
$$

where $a_{k}$ is a rational integer and

$$
V_{k}:=\sum_{n=n_{k}+k+1}^{\infty} \frac{\theta(n)}{q^{n}} .
$$

By (2.5) and (2.6) we have

$$
\begin{equation*}
n_{k} \leq M_{k} A H_{k}+\eta_{k} \leq 2^{k^{4}+k^{3}}+2^{k^{3}} \leq 2^{2 k^{4}} . \tag{2.11}
\end{equation*}
$$

Moreover, by the condition $\left(H_{2}\right)$ with $a=b=1$, we get

$$
\begin{equation*}
|\theta(n)| \leq n(2+\log n)^{\nu} \leq n^{2} \tag{2.12}
\end{equation*}
$$

for large $n$. Hence, by (2.9), (2.11), and (2.12)

$$
\begin{aligned}
\left|V_{k}\right| & \leq \sum_{n=n_{k}+k+1}^{n_{k}+2 k^{5}} \frac{|\theta(n)|}{|q|^{n}}+\sum_{n=n_{k}+2 k^{5}+1}^{\infty} \frac{|\theta(n)|}{|q|^{n}} \\
& \leq \frac{1}{|q|^{n_{k}+k}} \sum_{m=k+1}^{2 k^{5}}\left|\theta\left(n_{k}+m\right)\right|+\frac{1}{|q|^{n_{k}+2 k^{5}}} \sum_{m=1}^{\infty} \frac{\left(n_{k}+2 k^{5}+m\right)^{2}}{|q|^{n}} \\
& \leq \frac{k^{4 \nu+6}}{|q|^{n_{k}+k}}+\frac{16 c k^{10} 2^{4 k^{4}}}{|q|^{n_{k}+2 k^{5}}}
\end{aligned}
$$

where $c:=\sum_{n=1}^{\infty} n^{2} /|q|^{n}$. Thus, we obtain

$$
\begin{equation*}
\lim _{k \rightarrow \infty}\left|q^{n_{k}} V_{k}\right|=0 \tag{2.13}
\end{equation*}
$$

since $|q|>1$. Now we use the assumption that $f(q)$ is a rational number. Then there exist rational integers $\alpha$ and $\beta(\beta>0)$ such that $\beta f(q)+\alpha=0$. After multiplication by $q^{n_{k}-k}$, we can write by (2.10)

$$
\begin{equation*}
I_{k}:=\beta a_{k}+\alpha q^{n_{k}-k}=-\beta \frac{\theta\left(n_{k}\right)}{q^{k}}-\beta q^{n_{k}-k} V_{k} \tag{2.14}
\end{equation*}
$$

Multiplying (2.14) by $q^{k}$ yields

$$
\begin{equation*}
J_{k}:=\beta a_{k} q^{k}+\alpha q^{n_{k}}+\beta \theta\left(n_{k}\right)=-\beta q^{n_{k}} V_{k} \tag{2.15}
\end{equation*}
$$

It is clear that $I_{k}$ and $J_{k}$ are rational integers. By (2.9) we have

$$
\left|I_{k}\right| \leq \beta \frac{k^{4 \nu+6}}{|q|^{k}}+\frac{\beta}{|q|^{k}}\left|q^{n_{k}} V_{k}\right|, \quad\left|J_{k}\right| \leq \beta\left|q^{n_{k}} V_{k}\right|
$$

and hence by (2.13)

$$
\lim _{k \rightarrow \infty}\left|I_{k}\right|=\lim _{k \rightarrow \infty}\left|J_{k}\right|=0
$$

This implies that $I_{k}=J_{k}=0$ for every large $k$, since $I_{k}$ and $J_{k}$ are rational integers. Therefore by (2.14) and (2.15) we obtain

$$
\theta\left(n_{k}\right)=\frac{1}{\beta}\left(J_{k}-q^{k} I_{k}\right)=0
$$

for every large $k$, which is (1.7) as desired. The proof of Theorem 1.1 is completed.

## 3 Proof of Theorem 1.2

Suppose that the numbers (1.9) are linearly dependent over $\mathbb{Q}$; namely, there exist $\xi_{i, j} \in \mathbb{Z}(1 \leq i \leq \ell, 1 \leq$ $j \leq h)$, not all zero, such that

$$
\sum_{i=1}^{\ell} \sum_{j=1}^{h} \xi_{i, j} \sum_{n=1}^{\infty} \frac{\theta_{i}(n)}{q^{j n}}=\sum_{n=1}^{\infty} \frac{\Theta(n)}{q^{n}}
$$

is a rational number, where

$$
\begin{equation*}
\Theta(n):=\sum_{i=1}^{\ell} \sum_{j=1}^{h} \xi_{i, j} s_{i, j}(n) \tag{3.1}
\end{equation*}
$$

and

$$
s_{i, j}(n):= \begin{cases}\theta_{i}(n / j) & \text { if } n \text { is divisible by } j  \tag{3.2}\\ 0 & \text { Otherwise }\end{cases}
$$

Let $r(1 \leq r \leq h)$ be the least integer such that $\xi_{i, r} \neq 0$ for some $i$. Let $A$ and $B$ be any positive integers with $(A, h!)=1$. In what follows, we prove Theorem 1.2 by showing that there exist infinitely many positive multiples $N$ of $r$ such that

$$
\left\{\begin{array}{l}
\xi_{1, r} \theta_{1}(N / r)+\xi_{2, r} \theta_{2}(N / r)+\cdots+\xi_{\ell, r} \theta_{\ell}(N / r)=0  \tag{3.3}\\
N / r \equiv B \quad(\bmod A)
\end{array}\right.
$$

We apply Theorem 1.1 with $\theta:=\Theta$. To do this, we confirm that $\Theta$ satisfies the conditions $\left(H_{1}\right)$ and $\left(H_{2}\right)$ in Theorem 1.1. By the assumption on the $\theta_{i}$ 's, there exists a sequence $E \in \mathcal{E}$ and a positive integer $\gamma$ such that the condition $\left(H_{1}\right)$ is satisfied for all $\theta_{i}$ 's. Let $L:=\operatorname{lcm}(1,2, \ldots, h)$ and $\delta$ be the least positive integer such that $\operatorname{gcd}\left(e_{k}, L\right)=1$ for every $e_{k} \geq \delta$. Assume that the integer $n$ has the form $n=\left(e_{i_{1}} e_{i_{2}} \cdots e_{i_{m}}\right)^{\gamma} N$ with $\operatorname{gcd}\left(e_{i_{k}}, N\right)=1$, where $e_{i_{1}}, e_{i_{2}}, \ldots, e_{i_{m}} \geq \delta$ are distinct integers in $E$. If $n$ is not divisible by $j$, then $s_{i, j}(n)=0$. Otherwise, noting that $\operatorname{gcd}\left(e_{i_{k}}, j\right)=1$, we have $n / j=\left(e_{i_{1}} e_{i_{2}} \cdots e_{i_{m}}\right)^{\gamma}(N / j)$ with $\operatorname{gcd}\left(e_{i_{k}}, N / j\right)=1$. Then, by the condition $\left(H_{1}\right)$ for $\theta_{i}$ 's, each integer $\theta_{i}(n / j)$ is divisible by $q^{m}$, and so is $s_{i, j}(n)$ by (3.2). Thus, in any case, $s_{i, j}(n)$ is divisible by $q^{m}$, and hence we find by (3.1) that $\Theta$ satisfies $\left(H_{1}\right)$. Next we observe $\left(H_{2}\right)$ for $\Theta$. Let $a$ and $b$ be coprime positive integers. Suppose that the integer $a k+b$ is divisible by $j$. Then there exists a unique integer $k_{j}$ in the range $0 \leq k_{j}<j$ such that $k \equiv k_{j}(\bmod j)$, since the integers $a$ and $j$ are coprime because so are $a$ and $b$. Then for $n \geq \max \{a, b\}$ we obtain by the condition $\left(\mathrm{H}_{2}\right)$

$$
\begin{align*}
\sum_{k=0}^{n}\left|s_{i, j}(a k+b)\right| & =\sum_{\substack{k=0 \\
j \mid a k+b}}^{n}\left|\theta_{i}\left(\frac{a k+b}{j}\right)\right| \leq \sum_{k=0}^{\lfloor n / j\rfloor}\left|\theta_{i}\left(\frac{a\left(j k+k_{j}\right)+b}{j}\right)\right| \\
& \leq \sum_{k=0}^{2 n}\left|\theta_{i}\left(a k+\frac{a k_{j}+b}{j}\right)\right| \\
& \leq 2 n(2+\log 2 n)^{\nu_{0}} \tag{3.5}
\end{align*}
$$

for some positive constant $\nu_{0}$ depending on $\theta_{i}$ 's, where we used at the final inequality that the two integers $a$ and $\left(a k_{j}+b\right) / j$ are coprime and

$$
\max \left\{a, \frac{a k_{j}+b}{j}\right\}<a+b \leq 2 n
$$

Thus, using (3.1), (3.2), and (3.5), we obtain

$$
\sum_{k=0}^{n}|\Theta(a k+b)| \leq \sum_{i=1}^{\ell} \sum_{j=r}^{h}\left|\xi_{i, j}\right| \sum_{k=0}^{n}\left|s_{i, j}(a k+b)\right| \leq 2 h \ell \xi n(2+\log 2 n)^{\nu_{0}} \leq n(2+\log n)^{\nu}
$$

for $n \geq \max \{a, b\}$, where $\nu:=2 \nu_{0}+2 h \ell \xi$ and $\xi:=\max \left\{\left|\xi_{i, j}\right|: 1 \leq i \leq \ell, r \leq j \leq h\right\}$. Hence, $\Theta$ also satisfies $\left(H_{2}\right)$.
Let $C(0 \leq C<A)$ be an integer such that $1+h!C \equiv B(\bmod A)$. By the argument above, we can apply Theorem 1.1 with $\theta:=\Theta$. Then there exist infinitely many positive integers $N$ such that

$$
\left\{\begin{array}{l}
\Theta(N)=0  \tag{3.6}\\
N \equiv r(1+h!C) \quad(\bmod h!A)
\end{array}\right.
$$

The congruence (3.7) implies that the integer $N$ is a multiple of $r$, but not of $r+1, \ldots, h$. Hence, by (3.2) we have $s_{i, j}(N)=0$ for any $i, j$ with $r<j \leq h$. Thus, by (3.1), (3.2), and (3.6)

$$
0=\Theta(N)=\sum_{i=1}^{\ell} \xi_{i, r} s_{i, r}(N)=\sum_{i=1}^{\ell} \xi_{i, r} \theta_{i}(N / r)
$$

which gives (3.3). Moreover, the congruence (3.4) follows by (3.7), since $n=N / r \equiv 1+h!C \equiv B$ $(\bmod A)$. Theorem 1.2 is proved.

## 4 Proofs of Corollaries 1.1 and 1.2

We first give a sufficient condition for an arithmetic function $\theta$ to satisfy the condition $\left(H_{2}\right)$ in Theorem 1.1. Recall that $d(n)$ is the divisor function, defined in (1.3).

Lemma 4.1. Let $\theta$ be an arithmetic function. Assume that there exists a positive constant $\kappa$ such that

$$
\begin{equation*}
|\theta(n)| \leq(2+\log n)^{\kappa} d(n), \quad n \geq 1 \tag{4.1}
\end{equation*}
$$

Then $\theta$ satisfies the condition $\left(H_{2}\right)$ in Theorem 1.1.
Proof. Let $a$ and $b$ be arbitrary coprime positive integers. If $n \geq \max \{a, b\}$, then we have

$$
\sum_{i=0}^{n} d(a i+b) \leq 2 \sum_{i=0}^{n} \sum_{\substack{d \leq \sqrt{a i+b} \\ d \mid a i+b}} 1 \leq 2 \sum_{d \leq \sqrt{n^{2}+n}}\left(1+\left\lfloor\frac{n}{d}\right\rfloor\right) \leq 4 n(2+\log n)
$$

so that by (4.1)

$$
\begin{aligned}
\sum_{i=0}^{n}|\theta(a i+b)| & \leq \sum_{i=0}^{n}(2+\log (a i+b))^{\kappa} d(a i+b) \\
& \leq 2^{\kappa}(2+\log n)^{\kappa} \sum_{i=0}^{n} d(a i+b) \\
& \leq n(2+\log n)^{2 \kappa+3}
\end{aligned}
$$

The proof of Lemma 4.1 is completed.
Proof of Corollary 1.1. Clearly $\theta(n):=d(n) a_{n}$ satisfies the condition $\left(H_{1}\right)$ for the sequence of prime numbers and $\gamma:=|q|-1$. Moreover, $\theta(n)$ satisfies $\left(H_{2}\right)$ by Lemma 4.1 and therefore Theorem 1.2 applies, which proves Corollary 1.1.

Next we prove Corollary 1.2. In what follows, let $E:=\left\{e_{n}\right\}_{n \geq 1} \in \mathcal{E}$ and $F_{s}:=F_{s}(E)$ be as defined in (1.11). Define

$$
\begin{equation*}
a_{i}(n):=\sum_{x^{i} \mid n, x \in F_{s}} 1, \quad i=1,2, \ldots, \ell \tag{4.2}
\end{equation*}
$$

By definition (4.2), we have

$$
\begin{equation*}
a_{i}(n)=\prod_{j=1}^{m}\left(1+\left\lfloor\sigma_{j} / i\right\rfloor\right), \quad i=1,2, \ldots, \ell \tag{4.3}
\end{equation*}
$$

for the integer $n=e_{i_{1}}^{\sigma_{1}} e_{i_{2}}^{\sigma_{2}} \cdots e_{i_{m}}^{\sigma_{m}} \in F_{s}$ (i.e., $\sigma_{j} \leq s-1$ for any $j$ ) with distinct integers $e_{i_{1}}, e_{i_{2}}, \ldots, e_{i_{m}}$ in $E$, and

$$
\begin{equation*}
a_{i}(m n)=a_{i}(m) a_{i}(n), \quad i=1,2, \ldots, \ell, \tag{4.4}
\end{equation*}
$$

for coprime positive integers $m, n$ such that at least one of $m$ and $n$ belongs to $F_{s}$.
Proof of Corollary 1.2. We use the expressions

$$
\alpha_{i}(q):=\sum_{n \in F_{s}} \frac{1}{q^{n^{i}}-1}=\sum_{n=1}^{\infty} \frac{t_{i}(n)}{q^{n}-1}=\sum_{n=1}^{\infty} \frac{a_{i}(n)}{q^{n}}, \quad i=1,2, \ldots, \ell,
$$

where $t_{i}(n):=1$ if $n=y^{i}$ with $y \in F_{s},:=0$ otherwise, and $a_{i}(n)$ is as defined in (4.2). Then the arithmetic functions $a_{i}$ satisfy $\left(H_{1}\right)$ for the above $E=\left\{e_{n}\right\}_{\geq 1}$ and $\gamma:=|q| L-1$, where $L:=\operatorname{lcm}(1,2, \ldots, \ell)$. Indeed, assuming (1.5) for large distinct integers $e_{i_{1}}, e_{i_{2}}, \ldots, e_{i_{m}}$ in $E$, we have by (4.3) and (4.4)

$$
a_{i}(n)=a_{i}\left(\left(e_{i_{1}} e_{i_{2}} \cdots e_{i_{m}}\right)^{\gamma}\right) a_{i}(N)=\left(1+\left\lfloor|q| \frac{L}{i}-\frac{1}{i}\right\rfloor\right)^{m} a_{i}(N)=|q|^{m}(L / i)^{m} a_{i}(N),
$$

where we used the assumption $\gamma=|q| L-1 \leq s-1$ at the second equality. Hence, $a_{i}(n)$ is divisible by $q^{m}$ and the arithmetic functions $a_{i}$ 's satisfy the condition $\left(H_{1}\right)$. Moreover, by definition (4.2) we have $\left|a_{i}(n)\right| \leq d(n)$ for every $i=1,2, \ldots, \ell$, so that $a_{i}$ 's satisfy $\left(H_{2}\right)$ by Lemma 4.1
Suppose to the contrary that the $h \ell+1$ numbers (1.12)

$$
\text { 1, } \quad \alpha_{i}\left(q^{j}\right)=\sum_{n \in F_{s}} \frac{1}{q^{j n^{i}}-1}=\sum_{n=1}^{\infty} \frac{a_{i}(n)}{q^{j n}} \quad(i=1,2, \ldots, \ell, j=1,2, \ldots, h)
$$

are linearly dependent over $\mathbb{Q}$. Then, applying Theorem 1.2 with $\theta_{i}:=a_{i}$, we find that there exist integers $\xi_{r}, \xi_{r+1}, \ldots, \xi_{\ell}$ with $\xi_{r} \neq 0$ such that the following property holds; For any positive integer pair $A, B$ with $\operatorname{gcd}(A, h!)=1$, there exist a positive integer $n$ such that

$$
\left\{\begin{array}{l}
\xi_{r} a_{r}(n)+\xi_{r+1} a_{r+1}(n)+\cdots+\xi_{\ell} a_{\ell}(n)=0,  \tag{4.5}\\
n \equiv B \quad(\bmod A),
\end{array}\right.
$$

where we note that the integers $\xi_{i}$ 's are independent of $A$ and $B$. Let $\xi:=\max _{i \geq r}\left|\xi_{i}\right|$ and $k$ be a positive integer with $2^{k}\left|\xi_{r}\right|>\ell \xi$. Now we consider (4.5) and (4.6) for the integers

$$
\begin{equation*}
A:=\left(e_{u+1} e_{u+2} \cdots e_{u+k}\right)^{r+1}, \quad B:=\left(e_{u+1} e_{u+2} \cdots e_{u+k}\right)^{r} \tag{4.7}
\end{equation*}
$$

where $u$ is the least integer such that $\operatorname{gcd}\left(e_{m}, h!\right)=1$ holds for every $m>u$. Clearly, $\operatorname{gcd}(A, h!)=1$. Moreover, by (4.6) and (4.7) we have the form $n=\left(e_{u+1} e_{u+2} \cdots e_{u+k}\right)^{r} M$ with $\operatorname{gcd}\left(e_{v}, M\right)=1$. Noting that $r \leq \ell \leq|q| L-1 \leq s-1$, we obtain by (4.3) and (4.4)

$$
a_{i}(n)=\left\{\begin{aligned}
2^{k} a_{i}(M) & \text { if } i=r, \\
a_{i}(M) & \text { if } i>r .
\end{aligned}\right.
$$

Since $a_{r}(M) \geq a_{i}(M) \geq 1$ for every $i=r+1, \ldots, h$, we have by (4.5)

$$
2^{k}\left|\xi_{r}\right| \cdot a_{r}(M)=\left|\xi_{r}\right| a_{r}(n)=\left|\xi_{r+1} a_{r+1}(n)+\cdots+\xi_{\ell} a_{\ell}(n)\right| \leq \ell \xi \cdot a_{r}(M)
$$

so that $2^{k}\left|\xi_{r}\right| \leq \ell \xi$. This is a contradiction and the proof of the first assertion of Corollary 1.2 is completed.

The second assertion follows from the identities

$$
\sum_{n \in F_{s}} \frac{1}{q^{j n^{i}}+1}=\alpha_{i}\left(q^{j}\right)-2 \alpha_{i}\left(q^{2 j}\right), \quad i, j=1,2, \ldots
$$

and the first assertion shown above.
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