IRRATIONALITY EXPONENTS OF NUMBERS RELATED WITH CAHEN'S CONSTANT

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Abstract. We give lower and upper bounds of the irrationality exponent of general continued fractions satisfying certain conditions. Using it we estimate the irrationality exponents of continued fractions representing numbers related with Cahen's constant and deduce their transcendence from Roth's theorem.

1. Introduction

For a real number α, the irrationality exponent $\mu(\alpha)$ is defined by the infimum of the set of numbers μ for which the inequality

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^{\mu}} \tag{1.1}$$

has only finitely many rational solutions p/q, or equivalently the supremum of the set of numbers μ for which the inequality (1.1) has infinitely many solutions. If α is irrational, then $\mu(\alpha) \geq 2$. If α is a real algebraic irrationality, then $\mu(\alpha) = 2$ by Roth's theorem [8]. If $\mu(\alpha) = \infty$, then α is called a Liouville number.

The main theorem of this paper, Theorem 2 in Section 2, gives lower and upper bounds for the irrationality exponents $\mu(\alpha)$ of continued fractions

$$\alpha = \frac{a_1}{b_1 +} \frac{a_2}{b_2 +} \frac{a_3}{b_3 +} \cdots,$$

where a_n and b_n are nonzero integers satisfying certain conditions. We apply Theorem 2 to continued fractions representing numbers related to Cahen's constant and deduce their transcendence from the obtained lower bounds of their irrationality exponents.

In 1880 Sylvester [11] proved that any real number $0 < x < 1$ can be expanded uniquely in the series

$$x = \sum_{n=0}^{+\infty} \frac{1}{t_n},$$

where the t_n are integers satisfying the condition $t_0 \geq 2$, $t_{n+1} \geq t_n^2 - t_n + 1$ $(n \geq 0)$, and furthermore that x is rational if and only if the equality holds for all large n. He examined some of the properties of the (Sylvester) sequence $\{S_n\}_{n\geq 0}$ defined by

$$S_0 = 2, \quad S_{n+1} = S_n^2 - S_n + 1 \quad (n \geq 0), \tag{1.2}$$

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which satisfies

$$\sum_{n=0}^{+\infty} \frac{1}{S_n} = \sum_{n=0}^{+\infty} \left(\frac{1}{S_n - 1} - \frac{1}{S_{n+1} - 1} \right) = \frac{1}{S_0 - 1} = 1. \tag{1.3}$$

Cahen [2] and Sierpinski [9] independently obtained similar results for alternating series; namely, any irrational number $0 < x < 1$ can be uniquely written in the form

$$x = \sum_{n=0}^{+\infty} \frac{(-1)^n}{u_n},$$

where the u_n are integers satisfying $u_0 \geq 1$, $u_{n+1} \geq u_n^2 + u_n \ (n \geq 0)$. As an example, Cahen [2] mentioned that (Cahen's constant)

$$\sum_{n=0}^{+\infty} \frac{(-1)^n}{u_n} = \frac{1}{1} - \frac{1}{2} + \frac{1}{6} - \frac{1}{42} + \frac{1}{1806} - \frac{1}{3263442} + \cdots$$

is an irrational number, where $u_0 = 1$, $u_{n+1} = u_n^2 + u_n \ (n \geq 0)$, and hence $u_n = S_n - 1 \ (n \geq 0)$. We note that the sequence $\{s_n\}_{n \geq 0}$ defined by

$$s_0 = 2, \quad s_{n+1} = s_n^2 + s_n - 1 \quad (n \geq 0) \tag{1.4}$$

satisfies

$$\sum_{n=0}^{+\infty} \frac{(-1)^n}{s_n} = \sum_{n=0}^{+\infty} \left(\frac{(-1)^n}{s_n + 1} - \frac{(-1)^{n+1}}{s_{n+1} + 1} \right) = \frac{1}{s_0 + 1} = \frac{1}{3}. \tag{1.5}$$

In 1991 Davison and Shallit [4] proved the transcendence of Cahen's constant. Becker [1] improved the result by Mahler's method.

In this paper we generalize the sequences S_n and s_n defined in (1.2) and (1.4) by introducing the sequences $u_n = u_n(\varepsilon)$ satisfying $u_0 \in \mathbb{N}$, $u_0 > \max(1, \varepsilon)$, and the recurrence

$$u_{n+1} = u_n^2 - \varepsilon u_n + \varepsilon \quad (n \geq 0), \tag{1.6}$$

where ε is a non-zero integer given arbitrarily. Next, we define the numbers $\gamma_{l,\varepsilon} = \gamma_{l,\varepsilon}(u_0)$ by

$$\gamma_{l,\varepsilon} = \sum_{n=0}^{+\infty} (-1)^n \left(\frac{\varepsilon^n}{u_n - \varepsilon} \right)^l \quad (l = 1, 2, 3, \cdots). \tag{1.7}$$

We expand the numbers $\gamma_{l,\varepsilon}$ in continued fractions whose partial numerators a_n and denominators b_n satisfy the assumptions in Theorem 2, which will be formulated in section 2. Applying Theorem 2, we obtain the following

Theorem 1. *Let $\gamma_{l,\varepsilon}$ be the numbers defined by (1.7). Assume that u_0 and ε are coprime. Then $\mu(\gamma_{1,\varepsilon}) = 3$ and*

$$\begin{cases} 2 + \dfrac{2}{5} \leq \mu(\gamma_{2,\varepsilon}) \leq 2 + \dfrac{4}{7}, \\ 2 + \dfrac{2}{3l - 1} \leq \mu(\gamma_{l,\varepsilon}) \leq 2 + \dfrac{3(l-1)}{3l+1} \quad (l \geq 3). \end{cases} \tag{1.8}$$

Corollary 1. *For every positive integer l, $\gamma_{l,\varepsilon}$ is a non-Liouville transcendental number.*

Corollary 2. *Assume that* u_n *satisfies (1.6). Define*

$$\xi_\varepsilon = \xi_\varepsilon (u_0) = \sum_{n=0}^{+\infty} \frac{(-\varepsilon)^n}{u_n}. \tag{1.9}$$

Then $\mu(\xi_\varepsilon) = 3$ *and consequently* ξ_ε *is a non-Liouville transcendental number.*

Proof. It rests on a formula which generalizes (1.3) and (1.5). We have

$$\frac{1}{u_n - \varepsilon} - \frac{\varepsilon}{u_{n+1} - \varepsilon} = \frac{1}{u_n - \varepsilon} - \frac{\varepsilon}{u_n (u_n - \varepsilon)} = \frac{1}{u_n},$$

which yields

$$\sum_{n=0}^{+\infty} \frac{\varepsilon^n}{u_n} = \sum_{n=0}^{+\infty} \left(\frac{\varepsilon^n}{u_n - \varepsilon} - \frac{\varepsilon^{n+1}}{u_{n+1} - \varepsilon} \right) = \frac{1}{u_0 - \varepsilon} \in \mathbb{Q}. \tag{1.10}$$

Similarly, we can write

$$\gamma_{1,\varepsilon} = \sum_{n=0}^{+\infty} (-1)^n \frac{\varepsilon^n}{u_n - \varepsilon} = \sum_{n=0}^{+\infty} \left(\frac{\varepsilon^{2n}}{u_{2n} - \varepsilon} - \frac{\varepsilon^{2n+1}}{u_{2n+1} - \varepsilon} \right) = \sum_{n=0}^{+\infty} \frac{\varepsilon^{2n}}{u_{2n}}. \tag{1.11}$$

Therefore by (1.9), (1.10) and (1.11)

$$\xi_\varepsilon = 2\gamma_{1,\varepsilon} - \frac{1}{u_0 - \varepsilon}, \tag{1.12}$$

which proves that $\mu(\xi_\varepsilon) = \mu(\gamma_{1,\varepsilon}) = 3$ by Lemma 1 (Section 3 below). □

We give some examples of the numbers $\gamma_{l,\varepsilon}$ and ξ_ε.

Example 1. When $\varepsilon = 1$ and $u_0 = 2$, we have

$$\gamma_{l,1}(2) = \sum_{n=0}^{+\infty} \frac{(-1)^n}{(S_n - 1)^l} \quad (l = 1, 2, 3, \cdots),$$

$$\xi_1(2) = \sum_{n=0}^{+\infty} \frac{(-1)^n}{S_n}.$$

In particular, $\gamma_{1,1}(2)$ is Cahen's constant.

Example 2. When $\varepsilon = -1$ and $u_0 = 2$, we obtain

$$\gamma_{l,-1}(2) = \sum_{n=0}^{+\infty} \frac{(-1)^n}{(s_n + 1)^l} \quad (l = 2, 4, 6, \cdots),$$

$$\gamma_{l,-1}(2) = \sum_{n=0}^{+\infty} \frac{1}{(s_n + 1)^l} \quad (l = 1, 3, 5, \cdots),$$

$$\xi_{-1}(2) = \sum_{n=0}^{+\infty} \frac{1}{s_n}.$$

Example 3. When $\varepsilon = 2$ and $u_0 = 3$, u_n is the n-th Fermat number:

$$u_n = F_n = 2^{2^n} + 1.$$

Therefore we have

$$\gamma_{l,2}(3) = \sum_{n=0}^{+\infty} (-1)^n \left(\frac{2^n}{F_n - 2} \right)^l = \sum_{n=0}^{+\infty} (-1)^n \left(\frac{2^n}{2^{2^n} - 1} \right)^l \quad (l = 1, 2, 3, \cdots),$$

$$\xi_2(3) = \sum_{n=0}^{+\infty} \frac{(-2)^n}{F_n}.$$

It should be noted that the irrationality exponent of the sum of the reciprocals of Fermat numbers is equal to 2 (see [3]).

Example 4. Denote by L_n the sequence of Lucas numbers. Define

$$v_n = L_{2^{n+1}} = \Phi^{2^{n+1}} + \Phi^{-2^{n+1}},$$

where $\Phi = \frac{1}{2}\left(1 + \sqrt{5}\right)$ is the Golden number. Then clearly $v_{n+1} = v_n^2 - 2$. If we put $u_n = v_n + 2$, we see that $u_0 = 5$ and

$$u_{n+1} = u_n^2 - 4u_n + 4$$

for every $n \geq 0$. Therefore

$$\gamma_{l,4}(5) = \sum_{n=0}^{+\infty} (-1)^n \left(\frac{4^n}{L_{2^{n+1}} - 2} \right)^l \quad (l = 1, 2, 3, \cdots),$$

$$\xi_4(5) = \sum_{n=0}^{+\infty} \frac{(-4)^n}{L_{2^{n+1}}}.$$

The paper is organized as follows. In section 2 we state Theorem 2, which gives lower and upper bound for the irrationality exponent of general continued fractions under certain conditions. Section 3 is devoted to the proof of Theorem 2. In section 4, we prove Theorem 1 above by using Theorem 2. Finally, in section 5 we give an alternative proof of a special case of Theorem 1, namely $\mu(\gamma_{1,\varepsilon}) = 3$, by using approximations by the truncated sums of its defining series (1.7) in place of convergents of some continued fraction expansion. This proof rests heavily on formula (1.11), and for this reason it doesn't allow to estimate $\mu(\gamma_{l,\varepsilon})$ for $l \geq 2$.

2. Irrationality exponents for general continued fractions

We employ the usual notations for continued fractions :

$$b_0 + \frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots + \frac{a_n}{b_n} = b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \ddots + \cfrac{a_n}{b_n}}} = \frac{A_n}{B_n},$$

and

$$b_0 + \frac{a_1}{b_1} + \frac{a_2}{b_2} + \frac{a_3}{b_3} + \cdots = \lim_{n \to +\infty} \frac{A_n}{B_n},$$

where $\{A_n\}$ and $\{B_n\}$ are defined by

$$\begin{cases} A_{-1} = 1, \quad A_0 = b_0, \quad B_{-1} = 0, \quad B_0 = 1, \\ A_n = b_n A_{n-1} + a_n A_{n-2} \quad (n \geq 1), \\ B_n = b_n B_{n-1} + a_n B_{n-2} \quad (n \geq 1). \end{cases} \tag{2.1}$$

For complex numbers $\{a_n\}$ and $\{b_n\}$ with $a_n \neq 0$ for all $n \geq 1$, the infinite continued fraction written above is said to be convergent if at most a finite number of B_n vanish and if the limit exists. We refer to [5] or [7] for basic formulas and properties of continued fractions.

Theorem 2. *Let an infinite continued fraction*

$$\alpha = \frac{a_1}{b_1} + \frac{a_2}{b_2} + \frac{a_3}{b_3} + \cdots \tag{2.2}$$

be convergent, where a_n, b_n $(n \geq 1)$ are non zero rational integers. Assume that

$$\sum_{n=1}^{+\infty} \left| \frac{a_{n+1}}{b_n b_{n+1}} \right| < \infty, \tag{2.3}$$

and

$$\lim_{n\to+\infty} \left| \frac{a_1 a_2 \cdots a_n}{b_1 b_2 \cdots b_n} \right| = 0. \tag{2.4}$$

Then α is irrational and its irrationality exponent $\mu(\alpha)$ satisfies

$$2 + \sigma \leq \mu(\alpha) \leq 2 + \max(\tau_1, \tau_2), \tag{2.5}$$

where

$$\sigma = \limsup_{n\to+\infty} \frac{\log|b_{n+1}| - \log|a_1 a_2 \cdots a_{n+1}|}{\log|b_1 b_2 \cdots b_n|}, \tag{2.6}$$

$$\tau_1 = \limsup_{n\to+\infty} \frac{\log|a_1 a_2 \cdots a_n|}{\log|b_1 b_2 \cdots b_n| - \log|a_1 a_2 \cdots a_n|}, \tag{2.7}$$

and

$$\tau_2 = \limsup_{n\to+\infty} \frac{\log|b_{n+1}| - \log|a_1 a_2 \cdots a_{n+1}| + 2\log(A_n, B_n)}{\log|b_1 b_2 \cdots b_n| - \log|a_1 a_2 \cdots a_n|} \tag{2.8}$$

with (A_n, B_n) the greatest common divisor of A_n and B_n.

Remark 1. If we assume in Theorem 2 that $a_n > 0$ and $b_n > 0$ for every $n \geq 1$, (2.1) implies that

$$b_n < \frac{B_n}{B_{n-1}} = b_n + a_n \frac{B_{n-2}}{B_{n-1}} < b_n + \frac{a_n}{b_{n-1}},$$

and therefore by an easy induction

$$b_1 b_2 \cdots b_n < B_n < b_1 b_2 \cdots b_n \prod_{k=1}^{n} \left(1 + \frac{a_k}{b_k b_{k-1}}\right) < K b_1 b_2 \cdots b_n, \tag{2.9}$$

where $K = \prod_{k=1}^{\infty} (1 + a_k / b_k b_{k-1})$. Therefore the upper bound τ_1 given by (2.7) is the same, in this case, as the upper bound given by Lemma 2.3 in [6].

Corollary 3. *Let α be given in Theorem 2. If $\sigma > 0$, then α is a transcendental number.*

Theorem 2 with the formula

$$A_n B_{n-1} - A_{n-1} B_n = (-1)^{n-1} a_1 a_2 \cdots a_n \quad (n \geq 1) \tag{2.10}$$

leads to the following corollary, which will be proved at the end of Section 3.

Corollary 4. *Let an infinite continued fraction*

$$\alpha = \frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots \frac{a_n}{+ b_n}$$

be convergent, where a_n, b_n $(n \geq 1)$ are non-zero rational integers. Assume that

$$\sum_{n=1}^{+\infty} \left| \frac{a_{n+1}}{b_n b_{n+1}} \right| < \infty \tag{2.11}$$

and

$$\lim_{n \to +\infty} \frac{\log |a_n|}{\log |b_n|} = 0. \tag{2.12}$$

Then α is irrational and

$$\mu(\alpha) = 2 + \limsup_{n \to +\infty} \frac{\log |b_{n+1}|}{\log |b_1 b_2 \cdots b_n|}.$$

Remark 2. The irrationality exponent of an irrational number α with a simple continued fraction expansion

$$\alpha = [b_0; b_1, b_2, \cdots]$$

and convergents $p_n/q_n = [b_0; b_1, b_2, \cdots, b_n]$ is given by

$$\mu(\alpha) = 2 + \limsup_{n \to \infty} \frac{\log b_{n+1}}{\log q_n} \tag{2.13}$$

(cf. [10]). We note that, if b_n satisfies

$$\sum_{n=1}^{+\infty} \frac{1}{b_n b_{n+1}} < \infty,$$

then (2.13) becomes by using (2.9)

$$\mu(\alpha) = 2 + \limsup_{n \to \infty} \frac{\log b_{n+1}}{\log (b_1 b_2 \cdots b_n)}. \tag{2.14}$$

Hence Corollary 4 provides an extension of the formula (2.14) to a general continued fraction. Note also that the irrationality exponent for Cahen's constant $\gamma_{1,1}(2)$ could be computed from (2.14) by using the continued fraction expansion obtained by Davison and Shallit in [4]. But this is not the case for $\gamma_{l,\varepsilon}(u_0)$ if $(l, \varepsilon) \neq (1, 1)$.

3. Proof of Theorem 2

For the proof of Theorem 2 we need the following lemma, which is well known (see for example [6], Lemma 2.2). However, we will give here a self-contained proof, different from the proof in [6].

Lemma 1. *Let α be an irrational number. Then*

$$\mu(\alpha) = \mu\left(\frac{a + b\alpha}{c + d\alpha}\right) \tag{3.1}$$

for all integers a, b, c and d with $ad - bc \neq 0$.

Proof. Let α be a non-Liouville number. It is easily seen that

$$\mu(\alpha+m)=\mu(n\alpha)=\mu(\alpha) \tag{3.2}$$

for any rational integers m and $n \neq 0$. We prove that $\mu(1/\alpha)=\mu(\alpha)$ for $\alpha>0$. Suppose that (1.1) has infinitely many solutions p/q with $p,\, q>0$. We can assume that $\alpha/2<p/q<2\alpha$. Then

$$\left|\frac{1}{\alpha}-\frac{q}{p}\right|<\frac{C}{p^{\mu}}<\frac{1}{p^{\mu-\varepsilon}}$$

has infinitely many solutions q/p, where $C=(2\alpha)^{\mu-1}/\alpha$ and $\varepsilon>0$. This implies $\mu(1/\alpha)\geq\mu(\alpha)$. Replacing α by $1/\alpha$, we have $\mu(\alpha)\geq\mu(1/\alpha)$, and so $\mu(\alpha)=\mu(1/\alpha)$. Now, if $d=0$ we have by (3.2)

$$\mu\left(\frac{a+b\alpha}{c}\right)=\mu(a+b\alpha)=\mu(b\alpha)=\mu(\alpha)$$

and similarly for $d\neq 0$

$$\mu\left(\frac{a+b\alpha}{c+d\alpha}\right)=\mu\left(\frac{1}{d}\left(b+\frac{ad-bc}{c+d\alpha}\right)\right)=\mu\left(\frac{1}{c+d\alpha}\right)=\mu(c+d\alpha)=\mu(\alpha).$$

Hence the lemma follows if α is not a Liouville number. As a consequence, we see that α is not a Liouville number if and only if

$$\beta=\frac{a+b\alpha}{c+d\alpha}$$

is not a Liouville number. Therefore, if α is a Liouville number, then β is also a Liouville number and $\mu(\alpha)=\mu(\beta)=\infty$, which proves Lemma 1. □

Proof of Theorem 2. By the assumption (2.3), there is a positive integer N such that

$$\left|\frac{a_{n+1}}{b_n b_{n+1}}\right|\leq\frac{1}{4}\quad(n\geq N). \tag{3.3}$$

In the following, we assume that $N=1$ and the general case $N\geq 2$ will be discussed at the end of the proof. For any integers $n\geq 0$ and $k\geq 1$ we define

$$\begin{cases} A_{n,-1}=1, & A_{n,0}=0, \quad A_{n,k}=b_{n+k}A_{n,k-1}+a_{n+k}A_{n,k-2},\\ B_{n,-1}=0, & B_{n,0}=1, \quad B_{n,k}=b_{n+k}B_{n,k-1}+a_{n+k}B_{n,k-2},\end{cases}$$

so that

$$\frac{A_{n,k}}{B_{n,k}}=\frac{a_{n+1}}{b_{n+1}}+\frac{a_{n+2}}{b_{n+2}}+\cdots+\frac{a_{n+k}}{b_{n+k}},$$

and

$$\alpha_{n+1}=\frac{a_{n+1}}{b_{n+1}}+\frac{a_{n+2}}{b_{n+2}}+\frac{a_{n+3}}{b_{n+3}}+\cdots=\lim_{k\to+\infty}\frac{A_{n,k}}{B_{n,k}}. \tag{3.4}$$

In particular,

$$\begin{aligned}\alpha=\alpha_1&=\lim_{n\to+\infty}\frac{A_n}{B_n}\\ &=\frac{a_1}{b_1}+\frac{a_2}{b_2}+\cdots+\frac{a_{n-1}}{b_{n-1}}+\frac{a_n}{b_n+\alpha_{n+1}}=\frac{A_n+\alpha_{n+1}A_{n-1}}{B_n+\alpha_{n+1}B_{n-1}},\end{aligned} \tag{3.5}$$

where $A_n = A_{0,n}$ and $B_n = B_{0,n}$ $(n \geq 1)$. We have for $k \geq 2$

$$\begin{cases} A_{n,1} = a_{n+1}, & A_{n,k} = a_{n+1}b_{n+2}b_{n+3}\cdots b_{n+k}\theta_{n,1}\theta_{n,2}\cdots\theta_{n,k} \\ B_{n,1} = b_{n+1}, & B_{n,k} = b_{n+1}b_{n+2}b_{n+3}\cdots b_{n+k}\delta_{n,1}\delta_{n,2}\cdots\delta_{n,k} \end{cases} \tag{3.6}$$

with

$$\begin{cases} \theta_{n,1} = \theta_{n,2} = 1, \quad \theta_{n,k} = 1 + \dfrac{a_{n+k}}{b_{n+k-1}b_{n+k}\theta_{n,k-1}} \quad (k \geq 3)\,, \\ \delta_{n,1} = 1, \quad \delta_{n,k} = 1 + \dfrac{a_{n+k}}{b_{n+k-1}b_{n+k}\delta_{n,k-1}} \quad (k \geq 2)\,. \end{cases} \tag{3.7}$$

From (3.3) and (3.7) we deduce by induction on k that

$$\frac{1}{2} \leq \theta_{n,k} \leq 2, \quad \frac{1}{2} \leq \delta_{n,k} \leq 2 \quad (k \geq 1)\,. \tag{3.8}$$

We remark that $A_{n,k}B_{n,k} \neq 0$ for any $n \geq 0$ and $k \geq 1$ by (3.6) with (3.8). It follows from (3.4) and (3.6) that

$$\frac{b_{n+1}\alpha_{n+1}}{a_{n+1}} = \prod_{k=1}^{+\infty} \frac{\theta_{n,k}}{\delta_{n,k}} \quad (n \geq 0)\,,$$

where the infinite product converges to a non zero limit in view of (2.3), (3.7) and (3.8). As $\lim_{n\to+\infty} \theta_{n,k} = \lim_{n\to+\infty} \delta_{n,k} = 1$ uniformly with respect to k, we find

$$\lim_{n\to+\infty} \frac{b_{n+1}\alpha_{n+1}}{a_{n+1}} = 1,$$

particularly,

$$\frac{3}{4} \leq \frac{b_{n+1}\alpha_{n+1}}{a_{n+1}} \leq \frac{4}{3} \quad (n \geq n_0)\,. \tag{3.9}$$

Furthermore, since $B_n/B_{n-1} = b_n\delta_{0,n}$ by (3.6), we have by (3.3) and (3.8)

$$\left| \frac{a_{n+1}}{b_{n+1}} \frac{B_{n-1}}{B_n} \right| \leq \frac{1}{2} \quad (n \geq 1)\,. \tag{3.10}$$

Applying the formula (2.10), we deduce from (3.5)

$$\alpha - \frac{A_n}{B_n} = \frac{(-1)^n \alpha_{n+1}a_1a_2\cdots a_n}{B_n\left(B_n + \alpha_{n+1}B_{n-1}\right)} = \frac{(-1)^n a_1a_2\cdots a_{n+1}}{b_{n+1}B_n^2\left(\dfrac{a_{n+1}}{b_{n+1}\alpha_{n+1}} + \dfrac{a_{n+1}}{b_{n+1}}\dfrac{B_{n-1}}{B_n}\right)},$$

which together with (3.9) and (3.10) yields

$$\frac{1}{4}\frac{|a_1a_2\cdots a_{n+1}|}{|b_{n+1}|\,B_n^2} < \left|\alpha - \frac{A_n}{B_n}\right| < 4\frac{|a_1a_2\cdots a_{n+1}|}{|b_{n+1}|\,B_n^2} \quad (n \geq n_0)\,. \tag{3.11}$$

It follows from (3.6) that

$$\frac{2}{3}\rho\,|b_1b_2\cdots b_n| < |B_n| < \frac{3}{2}\rho\,|b_1b_2\cdots b_n| \quad (n \geq n_1 \geq n_0)\,, \tag{3.12}$$

where $\rho = \prod_{k=0}^{\infty} \delta_{0,k} > 0$. Combining (3.11) and (3.12), we obtain

$$\frac{1}{6\rho}\left|\frac{a_1a_2\cdots a_{n+1}}{b_1b_2\cdots b_{n+1}}\right| < |B_n\alpha - A_n| < \frac{6}{\rho}\left|\frac{a_1a_2\cdots a_{n+1}}{b_1b_2\cdots b_{n+1}}\right| \quad (n \geq n_1)\,. \tag{3.13}$$

Suppose that α is a rational number a/b with $b > 0$. Then we have by (3.13)

$$1 \leq |aB_n - bA_n| < \frac{6b}{\rho}\left|\frac{a_1a_2\cdots a_{n+1}}{b_1b_2\cdots b_{n+1}}\right| \quad (n \geq n_1)\,,$$

where the right-hand side tends to zero as $n \to \infty$ by the assumption (2.4), which is a contradiction. Hence α is irrational.

Let σ be defined in (2.6). We prove the lower bound for $\mu(\alpha)$ in (2.5). There is nothing to prove if $\sigma \leq 0$ and we can assume that $\sigma > 0$. Taking the logarithms in (3.11) yields

$$\log \left| \alpha - \frac{A_n}{B_n} \right| < \log 4 - \left(\log |b_{n+1}| - \log |a_1 a_2 \cdots a_{n+1}| \right) - 2 \log |B_n| \,. \tag{3.14}$$

As $\sigma > 0$, for $\varepsilon > 0$ sufficiently small there exist infinitely many n such that

$$\frac{\log |b_{n+1}| - \log |a_1 a_2 \cdots a_{n+1}|}{\log |b_1 b_2 \cdots b_n|} \geq \sigma - \varepsilon > 0. \tag{3.15}$$

Moreover, by the right-hand side of (3.12), we have for every n sufficiently large

$$\log |b_1 b_2 \cdots b_n| > \log |B_n| - \log \left(\frac{3\rho}{2} \right) > (1 - \varepsilon) \log |B_n| > 0. \tag{3.16}$$

Hence, by using (3.14), (3.15) and (3.16) we obtain for every $\varepsilon > 0$ and infinitely many n

$$\begin{aligned} \log \left| \alpha - \frac{A_n}{B_n} \right| &< \log 4 - (\sigma - \varepsilon)(1 - \varepsilon) \log |B_n| - 2 \log |B_n| \\ &< -\left(2 + \sigma - \varepsilon (\sigma + 2 - \varepsilon)\right) \log |B_n| \,. \end{aligned}$$

Therefore, for every $\varepsilon > 0$ sufficiently small there exist infinitely many n such that

$$\left| \alpha - \frac{A_n}{B_n} \right| < \frac{1}{B_n^{2+\sigma-\varepsilon(\sigma+2-\varepsilon)}},$$

which proves that $\mu(\alpha) \geq 2 + \sigma$.

We prove now the upper bound for $\mu(\alpha)$ in (2.5). Choose any rational number p/q. We may assume that p and q are coprime and

$$q > \frac{\rho}{12} \min_{n \geq n_1} \left| \frac{b_1 b_2 \cdots b_n}{a_1 a_2 \cdots a_n} \right|.$$

In view of (2.4) there exists $n = n(q) \geq n_1$ such that

$$\left| \frac{b_1 b_2 \cdots b_n}{a_1 a_2 \cdots a_n} \right| \leq \frac{12}{\rho} q < \left| \frac{b_1 b_2 \cdots b_{n+1}}{a_1 a_2 \cdots a_{n+1}} \right|. \tag{3.17}$$

We consider two cases:

Case 1. $A_n q - B_n p \neq 0$. Then $|A_n q - B_n p| \geq 1$. We have

$$B_n \left(\alpha - \frac{p}{q} \right) = \frac{A_n q - B_n p}{q} + B_n \alpha - A_n.$$

The right inequalities in (3.13) and (3.17) yield

$$|B_n \alpha - A_n| < \frac{1}{2q}.$$

Hence we get

$$\left| \alpha - \frac{p}{q} \right| > \frac{1}{2q |B_n|} = \frac{1}{2q^{1+\tau_{1,n}}}, \tag{3.18}$$

where

$$\tau_{1,n} = \frac{\log|B_n|}{\log q} < \frac{\log|b_1 b_2 \cdots b_n| + \log(3\rho/2)}{\log|b_1 b_2 \cdots b_n| - \log|a_1 a_2 \cdots a_n| + \log(\rho/12)}, \tag{3.19}$$

using the right and left inequalities in (3.12) and (3.17) respectively.

Case 2. $A_n q - B_n p = 0$. Then $A_n = (A_n, B_n)\, p$, $B_n = (A_n, B_n)\, q$, recalling that p and q are coprime. So we deduce from (3.11)

$$\left|\alpha - \frac{p}{q}\right| = \left|\alpha - \frac{A_n}{B_n}\right| > \frac{1}{4}\frac{|a_1 a_2 \cdots a_{n+1}|}{|b_{n+1}|\,(A_n, B_n)^2\, q^2} = \frac{1}{4}\frac{1}{q^{2+\tau_{2,n}}}, \tag{3.20}$$

where

$$\tau_{2,n} = \frac{\log|b_{n+1}| - \log|a_1 a_2 \cdots a_{n+1}| + 2\log(A_n, B_n)}{\log q}$$
$$< \frac{\log|b_{n+1}| - \log|a_1 a_2 \cdots a_{n+1}| + 2\log(A_n, B_n)}{\log|b_1 b_2 \cdots b_n| - \log|a_1 a_2 \cdots a_n| + \log(\rho/12)},$$

using the left inequality in (3.18). The upper bound

$$\mu(\alpha) \le 2 + \max(\tau_1, \tau_2)$$

is obtained from (3.18), (3.19) and (3.20), and the proof of Theorem 2 is completed in the case $N = 1$.

We assume finally that $N \ge 2$. We can apply Theorem 2 to α_N. Then

$$2 + \sigma \le \mu(\alpha_N) \le 2 + \max(\tau_1, \tau_2), \tag{3.21}$$

where

$$\sigma = \limsup_{n\to+\infty} \frac{\log|b_{N+n}| - \log|a_N a_{N+1} \cdots a_{N+n}|}{\log|b_N b_{N+1} \cdots b_{N+n-1}|},$$
$$\tau_1 = \limsup_{n\to+\infty} \frac{\log|a_N a_{N+1} \cdots a_{N+n-1}|}{\log|b_N b_{N+1} \cdots b_{N+n-1}| - \log|a_N a_{N+1} \cdots a_{N+n-1}|},$$
$$\tau_2 = \limsup_{n\to+\infty} \frac{\log|b_{N+n}| - \log|a_N a_{N+1} \cdots a_{N+n}| + 2\log(A_{N-1,n}, B_{N-1,n})}{\log|b_N b_{N+1} \cdots b_{N+n-1}| - \log|a_N a_{N+1} \cdots a_{N+n-1}|}.$$

However, we have for every $n \in \mathbb{N}$

$$\begin{cases} A_{N-1+n} = A_{N-2}A_{N-1,n} + A_{N-1}B_{N-1,n} \\ B_{N-1+n} = B_{N-2}A_{N-1,n} + B_{N-1}B_{N-1,n} \end{cases}$$

since the sequences A_{N-1+n}, $A_{N-1,n}$, B_{N-1+n} and $B_{N-1,n}$ satisfy the same recurrence relation and this is true for $n = -1$ and $n = 0$. Therefore

$$(A_{N-1,n}, B_{N-1,n}) \le (A_{N-1+n}, B_{N-1+n})$$

and the proof of Theorem 2 is complete because $\mu(\alpha_N) = \mu(\alpha)$ by (2.10), the last equality of (3.5), and Lemma 1.

Proof of Corollary 4. First we prove that the assumptions (2.11) and (2.12) imply (2.4). Indeed (2.4) holds trivially if $|a_n| = 1$ for all large n. Otherwise, we see by (2.11) that $\lim_{n\to+\infty} |b_1 b_2 \cdots b_n| = +\infty$, and so

$$\lim_{n\to+\infty} \log|b_1 b_2 \cdots b_n| = +\infty. \tag{3.22}$$

From (2.12) and (3.22) we can deduce that

$$\lim_{n\to+\infty} \frac{\log|a_1a_2\cdots a_n|}{\log|b_1b_2\cdots b_n|} = 0. \tag{3.23}$$

This implies in particular that $|a_1a_2\cdots a_n| < |b_1b_2\cdots b_n|^{\frac{1}{2}}$ for every large n, and hence (2.4) follows.

Therefore we can apply Theorem 2. By the definition of σ, we have

$$\sigma = \limsup_{n\to+\infty} \frac{\log|b_{n+1}|}{\log|b_1b_2\cdots b_n|}$$

using (2.12) and (3.23). Similarly we get

$$\tau_1 \leq \limsup_{n\to+\infty} \frac{1}{\dfrac{\log|b_1b_2\cdots b_n|}{\log|a_1a_2\cdots a_n|} - 1} = 0 \leq \sigma.$$

Finally, we compute τ_2. Applying the formula (2.10) to (A_n, B_n) in the definition of τ_2, we find

$$\tau_2 \leq \limsup_{n\to+\infty} \frac{\log|b_{n+1}| - \log|a_{n+1}| + \log|a_1a_2\cdots a_n|}{\log|b_1b_2\cdots b_n| - \log|a_1a_2\cdots a_n|} = \sigma,$$

and the proof is completed.

4. Proof of theorem 1

For proving theorem 1 we will need three lemmas.

Lemma 2. *Let ε be a non zero integer. Let $u_n = u_n(\varepsilon)$ be defined by $u_0 \in \mathbb{N}$, with $u_0 > \max(1, \varepsilon)$, and*

$$u_{n+1} = u_n^2 - \varepsilon u_n + \varepsilon \quad (n \geq 0). \tag{4.1}$$

Then u_n is a positive integer for every $n \geq 0$ and there exists a constant $\lambda > 0$ such that

$$\log u_n = \lambda 2^n + O(1). \tag{4.2}$$

Proof. First we prove by induction that

$$u_n > \max\left(2^{n-1} + 1, \varepsilon\right) \tag{4.3}$$

for every $n \geq 0$. This is true for $n = 0$ by hypothesis. Assume that it is true for a given $n \geq 0$. We have by (4.1)

$$u_{n+1} - \varepsilon = u_n(u_n - \varepsilon). \tag{4.4}$$

This implies $u_{n+1} > \varepsilon$, since $u_n > \max\left(2^{n-1} + 1, \varepsilon\right)$. Moreover

$$u_{n+1} - u_n = (u_n - \varepsilon)(u_n - 1) > 2^{n-1}$$

since by the induction hypothesis $u_n - \varepsilon \geq 1$ and $u_n > 2^{n-1} + 1$. Therefore

$$u_{n+1} > u_n + 2^{n-1} > 2^n + 1,$$

which proves (4.3). Hence the sequence u_n is a sequence of positive integers.

To prove (4.2), we observe that, for every positive integer k,

$$\log u_k = 2\log u_{k-1} + \log\left(1 - \frac{\varepsilon}{u_{k-1}} + \frac{\varepsilon}{u_{k-1}^2}\right).$$

Multiplying by 2^{n-k} and summing for k from 1 to n yields

$$\log u_n = 2^n \log u_0 + 2^n \sum_{k=1}^{n} \frac{1}{2^k} \log\left(1 - \frac{\varepsilon}{u_{k-1}} + \frac{\varepsilon}{u_{k-1}^2}\right).$$

Since $u_n > 2^{n-1} + 1$, the series

$$\sum_{k=1}^{+\infty} \frac{1}{2^k} \log\left(1 - \frac{\varepsilon}{u_{k-1}} + \frac{\varepsilon}{u_{k-1}^2}\right)$$

is convergent, and we can write

$$\log u_n = 2^n \left(\log u_0 + \sum_{k=1}^{+\infty} \frac{1}{2^k} \log\left(1 - \frac{\varepsilon}{u_{k-1}} + \frac{\varepsilon}{u_{k-1}^2}\right)\right) - \sum_{k=n+1}^{+\infty} \frac{1}{2^{k-n}} \log\left(1 - \frac{\varepsilon}{u_{k-1}} + \frac{\varepsilon}{u_{k-1}^2}\right).$$

Finally we have

$$\left|\sum_{k=n+1}^{+\infty} \frac{1}{2^{k-n}} \log\left(1 - \frac{\varepsilon}{u_{k-1}} + \frac{\varepsilon}{u_{k-1}^2}\right)\right| \le \sum_{k=n+1}^{+\infty} \frac{1}{2^{k-n}} \le 1$$

when $n \to +\infty$, which proves (4.2). □

Lemma 3. *Let x_1, x_2, $x_3 \cdots$, y_1, y_2, $y_3 \cdots$ be non zero complex numbers, and let*

$$X_l = \sum_{n=1}^{+\infty} (-1)^n \left(\frac{y_1 y_2 \cdots y_n}{x_1 x_2 \cdots x_n}\right)^l, \quad (l = 1, 2, 3, \cdots) \tag{4.5}$$

be convergent. Then

$$X_l = \frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots + \frac{a_n}{b_n} + \cdots,$$

where

$$\begin{cases} a_1 = -y_1^l, \quad a_n = y_n^l x_{n-1}^{l-1} \quad (n \ge 2), \\ b_1 = x_1^l, \quad b_2 = x_1^{-1}\left(x_2^l - y_2^l\right), \\ b_{2k} = \dfrac{x_2 x_4 \cdots x_{2k-2}}{x_1 x_3 \cdots x_{2k-1}} \left(x_{2k}^l - y_{2k}^l\right) \quad (k \ge 2), \\ b_{2k+1} = \dfrac{x_1 x_3 \cdots x_{2k-1}}{x_2 x_4 \cdots x_{2k}} \left(x_{2k+1}^l - y_{2k+1}^l\right) \quad (k \ge 1). \end{cases} \tag{4.6}$$

Proof. Applying Euler's transformation

$$\sum_{i=1}^{n} \rho_1 \rho_2 \cdots \rho_i = \frac{\rho_1}{1} + \frac{-\rho_2}{1+\rho_2} + \frac{-\rho_3}{1+\rho_3} + \cdots + \frac{-\rho_n}{1+\rho_n}$$

with $\rho_i = -y_i^l x_i^{-l}$ $(i \ge 1)$, we have

$$X_l = \frac{-y_1^l x_1^{-l}}{1} + \frac{y_2^l x_2^{-l}}{1 - y_2^l x_2^{-l}} + \frac{y_3^l x_3^{-l}}{1 - y_3^l x_3^{-l}} + \cdots + \frac{y_n^l x_n^{-l}}{1 - y_n^l x_n^{-l}} + \cdots.$$

Hence for any sequence t_n of non zero real numbers

$$X_l = \frac{-t_1\left(y_1^l x_1^{-l}\right)}{t_1} \; \frac{t_2 t_1\left(y_2^l x_2^{-l}\right)}{+\,t_2\left(1-y_2^l x_2^{-l}\right)} \; \frac{t_3 t_2\left(y_3^l x_3^{-l}\right)}{+\,t_3\left(1-y_3^l x_3^{-l}\right)+} \cdots \frac{t_n t_{n-1}\left(y_n^l x_n^{-l}\right)}{+\,t_n\left(1-y_n^l x_n^{-l}\right)+} \cdots . \tag{4.7}$$

We apply (4.7) with $t_1 = x_1^l$, $t_2 = x_1^{-1} x_2^l$, and

$$t_{2k} = \frac{x_2 x_4 \cdots x_{2k-2}}{x_1 x_3 \cdots x_{2k-1}} x_{2k}^l \quad (k \geq 2)\,, \quad t_{2k+1} = \frac{x_1 x_3 \cdots x_{2k-1}}{x_2 x_4 \cdots x_{2k}} x_{2k+1}^l \quad (k \geq 1)\,.$$

We observe that

$$t_{n+1} t_n = x_{n+1}^l x_n^{l-1} \quad (n \geq 1)\,.$$

Therefore from (4.7) we obtain

$$X_l = \frac{a_1}{b_1+} \frac{a_2}{b_2+} \frac{a_3}{b_3+} \cdots \frac{a_n}{+\,b_n+} \cdots , \tag{4.8}$$

where a_n and b_n are given by (4.6), which proves Lemma 3. □

Lemma 4. *Let u_n be defined in Lemma 2. Assume moreover that $(u_0, \varepsilon) = 1$. Then, for integers $m, n \geq 1$ and $0 \leq i < n$,*

$$(i) \qquad (u_n, u_i) = (u_n, \varepsilon) = 1,$$
$$(ii) \qquad u_n^m \equiv \varepsilon^m \pmod{u_i},$$
$$(iii) \qquad \frac{u_n^m - \varepsilon^m}{u_n - \varepsilon} \equiv m\varepsilon^{m-1} \pmod{u_i}.$$

Proof. We have $u_{i+1} \equiv \varepsilon \pmod{u_i}$ by (4.1), and an easy induction using again (4.1) shows that

$$u_n \equiv \varepsilon \pmod{u_i} \quad \text{if} \quad i < n,$$

which proves (*ii*) and (*iii*). For proving (*i*), assume that there exists a prime p which divides u_1 and ε. Then by (4.1) p divides $u_0\,(u_0 - \varepsilon)$ and p divides u_0, which is impossible since $(u_0, \varepsilon) = 1$. Hence $(u_0, \varepsilon) = 1$ implies $(u_1, \varepsilon) = 1$ and by induction $(u_n, \varepsilon) = 1$ for every $n \geq 0$. Now let $d = (u_n, u_i)$, $i < n$. By using (*ii*) we have $u_n \equiv q u_i + \varepsilon$ with $q \geq 0$, and therefore d divides u_i and ε, whence $d = 1$. □

Proof of Theorem 1. By using (4.4), we see that

$$u_n - \varepsilon = (u_1 - \varepsilon)\, u_1 u_2 \cdots u_{n-1} \quad (n \geq 2)\,. \tag{4.9}$$

By (4.9) we can write

$$\gamma_{l,\varepsilon} = \frac{1}{(u_0 - \varepsilon)^l} - \frac{\varepsilon^l}{(u_1 - \varepsilon)^l} \left(1 + \sum_{n=1}^{+\infty} (-1)^n \left(\frac{\varepsilon^n}{u_1 u_2 \cdots u_n}\right)^l\right). \tag{4.10}$$

Now we define

$$\gamma_{l,\varepsilon}^* = \sum_{n=1}^{+\infty} (-1)^n \left(\frac{\varepsilon^n}{u_1 u_2 \cdots u_n}\right)^l .$$

Then $\mu\left(\gamma_{l,\varepsilon}\right) = \mu\left(\gamma_{l,\varepsilon}^*\right)$ by (4.10) and Lemma 1. From Lemma 3 with $x_n = u_n$ and $y_n = \varepsilon$ we get

$$\gamma_{l,\varepsilon}^* = \frac{a_1}{b_1+} \frac{a_2}{b_2+} \cdots \frac{a_n}{+\,b_n+} \cdots \tag{4.11}$$

with

$$\left\{\begin{array}{l} a_1 = -\varepsilon^l, \quad a_n = \varepsilon^l u_{n-1}^{l-1} \quad (n \geq 2), \\ b_1 = u_1^l, \quad b_2 = u_1^{-1}\left(u_2^l - \varepsilon^l\right), \\ b_{2k} = \dfrac{u_2 u_4 \cdots u_{2k-2}}{u_1 u_3 \cdots u_{2k-1}}\left(u_{2k}^l - \varepsilon^l\right) \quad (k \geq 2), \\ b_{2k+1} = \dfrac{u_1 u_3 \cdots u_{2k-1}}{u_2 u_4 \cdots u_{2k}}\left(u_{2k+1}^l - \varepsilon^l\right) \quad (k \geq 1). \end{array}\right. \tag{4.12}$$

Hence we can write by (4.9)

$$\left\{\begin{array}{l} b_{2k} = (u_1 - \varepsilon)(u_2 u_4 \cdots u_{2k-2})^2 \dfrac{u_{2k}^l - \varepsilon^l}{u_{2k} - \varepsilon} \quad (k \geq 2), \\ b_{2k+1} = (u_1 - \varepsilon)(u_1 u_3 \cdots u_{2k-1})^2 \dfrac{u_{2k+1}^l - \varepsilon^l}{u_{2k+1} - \varepsilon} \quad (k \geq 1), \end{array}\right. \tag{4.13}$$

and therefore $a_n, b_n \in \mathbb{N}^0$ for every $n \geq 1$. Moreover by (4.13) we also observe that, for every $n \geq 3$,

$$u_{n-2k} \, | b_n \,, \quad 1 \leq k \leq \frac{1}{2}(n-1). \tag{4.14}$$

where $a \,| b$ denotes as usual that a divides b. Now we apply Theorem 2 to the continued fraction $\gamma_{l,\varepsilon}^*$ with a_n and b_n given by (4.12). If we take the logarithms in (4.12) and use (4.2), we see after some calculation that

$$\log |a_n| = \lambda (l-1) 2^{n-1} + O(1), \tag{4.15}$$

$$\log |a_1 a_2 \cdots a_n| = \lambda (l-1) 2^n + O(n), \tag{4.16}$$

$$\log b_n = \frac{1}{3}\lambda (3l-1) 2^n + O(n), \tag{4.17}$$

$$\log (b_1 b_2 \cdots b_n) = \frac{2}{3}\lambda (3l-1) 2^n + O\left(n^2\right). \tag{4.18}$$

Therefore a_n and b_n satisfy the assumptions (2.3) and (2.4). Using (4.16)-(4.18) in (2.6) and (2.7), we obtain

$$\sigma = \frac{2}{3l-1}, \quad \tau_1 = \frac{3(l-1)}{3l+1},$$

which already gives the lower bound for $\mu(\gamma_{l,\varepsilon})$ by using Theorem 2 and the exact value $\mu(\gamma_{1,\varepsilon}) = 3$ by using Corollary 4.

To estimate τ_2 defined by (2.8), we use the estimate

$$\log (A_n, B_n) = O\left(n^2\right) \tag{4.19}$$

as $n \to \infty$, where A_n, B_n are defined in (2.1) with a_n, b_n given in (4.12). This estimate will be proved later. Then from (2.8) with (4.16)-(4.19) we can deduce

$$\tau_2 = \frac{4}{3l+1},$$

and we see that $\tau_2 < \tau_1$ for $l \geq 3$, while $\tau_2 > \tau_1$ for $l = 2$. This yields the upper bound in (1.8).

It remains to prove (4.19). If $l = 1$, then $\log (A_n, B_n) = O(n)$, since

$$(A_n, B_n) \,\Big|\, \varepsilon^{nl} (u_0 u_1 \cdots u_{n-1})^{l-1} \quad (n \geq 1, \ l \geq 1) \tag{4.20}$$

by (2.10) and (4.12). So we can assume that $l \geq 2$. We define

$$d_i = d_i(n) = ((A_n, B_n), u_i) \quad (0 \leq i \leq n-1). \tag{4.21}$$

By Lemma 4 (i), we have $(d_i, d_j) = 1$ if $i \neq j$, and therefore $d_0 d_1 \cdots d_{n-1}$ divides (A_n, B_n). Put $(A_n, B_n) = k_n d_0 d_1 \cdots d_{n-1}$ and $u_i = d_i u_i'$ for $i = 0, 1, \cdots, n-1$. Then by (4.21) we see that $(k_n, u_i') = 1$ for $i = 0, 1, \cdots, n-1$. By (4.20) we have

$$k_n \left| \varepsilon^{nl} \left(d_0 d_1 \cdots d_{n-1}\right)^{l-1} \left(u_0' u_1' \cdots u_{n-1}'\right)^{l-1} \right. \quad (n \geq 1), \tag{4.22}$$

which implies that $k_n \left| \varepsilon^{nl} \left(d_0 d_1 \cdots d_{n-1}\right)^{l-1}\right.$, whence $(A_n, B_n) \left| \varepsilon^{nl} \left(d_0 d_1 \cdots d_{n-1}\right)^{l}\right.$. Hence to prove (4.19) it is enough to prove that

$$\log \left(d_1 d_2 \cdots d_{n-1}\right) = O\left(n^2\right), \tag{4.23}$$

since $d_0 \leq u_0$. For proving this, we will estimate d_j for every integer j such that

$$1 \leq j \leq n-1. \tag{4.24}$$

From (2.1) we have for every integer i such that $0 \leq i \leq n-2$

$$\begin{pmatrix} A_{n-i-1} \\ A_{n-i} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ a_{n-i} & b_{n-i} \end{pmatrix} \begin{pmatrix} A_{n-i-2} \\ A_{n-i-1} \end{pmatrix}, \tag{4.25}$$

$$\begin{pmatrix} A_{n-i-1} \\ A_{n-i} \end{pmatrix} = \begin{pmatrix} a_{n-i-1} & b_{n-i-1} \\ a_{n-i-1} b_{n-i} & a_{n-i} + b_{n-i} b_{n-i-1} \end{pmatrix} \begin{pmatrix} A_{n-i-3} \\ A_{n-i-2} \end{pmatrix}. \tag{4.26}$$

Define for every integer i such that $0 \leq i \leq \frac{1}{2}(n-2)$

$$N_i = \begin{pmatrix} a_{n-2i-1} & b_{n-2i-1} \\ a_{n-2i-1} b_{n-2i} & a_{n-2i} + b_{n-2i} b_{n-2i-1} \end{pmatrix}. \tag{4.27}$$

By (4.26) we can write for every integer k such that $0 \leq k \leq \frac{1}{2}(n-1)$

$$\begin{pmatrix} A_{n-1} \\ A_n \end{pmatrix} = \left(\prod_{i=0}^{k-1} N_i\right) \begin{pmatrix} A_{n-2k-1} \\ A_{n-2k} \end{pmatrix}. \tag{4.28}$$

Case 1. Assume that $j = n-2k$, with $1 \leq k \leq \frac{1}{2}(n-1)$.
Then by (4.12), Lemma 4 (ii) and (4.14) we have

$$N_i \equiv \begin{pmatrix} a_{n-2i-1} & b_{n-2i-1} \\ 0 & \varepsilon^{2l-1} \end{pmatrix} \quad (\operatorname{mod} u_{n-2k}). \tag{4.29}$$

Therefore by (4.28) we see that $A_n \equiv \varepsilon^{k(2l-1)} A_{n-2k} \ (\operatorname{mod} u_{n-2k})$, and the same holds for B_n. As $d_{n-2k} \left| u_{n-2k}\right.$ and $d_{n-2k} \left| (A_n, B_n)\right.$, this yields

$$d_{n-2k} \left| \varepsilon^{k(2l-1)} \left(A_{n-2k}, B_{n-2k}\right)\right. . \tag{4.30}$$

By (4.20), this implies that

$$d_{n-2k} \left| \varepsilon^{nl+k(2l-1)} \left(u_0 u_1 \cdots u_{n-2k-1}\right)^{l-1}\right. .$$

As $d_{n-2k} \left| u_{n-2k}\right.$, we have $(d_{n-2k}, u_i) = (d_{n-2k}, \varepsilon) = 1$ for $i < n-2k$ by Lemma 4 (i), and therefore

$$d_{n-2k} = 1, \quad 1 \leq k \leq \frac{1}{2}(n-1). \tag{4.31}$$

Case 2. Assume that $j = n-2k-1$, with $0 \leq k \leq \frac{1}{2}(n-2)$.

Then again by (4.12), Lemma 4 (*ii*) and (4.14) we have

$$N_i \equiv \varepsilon^{2l-1} \begin{pmatrix} 1 & 0 \\ b_{n-2i} & 1 \end{pmatrix} \pmod{u_{n-2k-1}}, \tag{4.32}$$

which yields

$$\prod_{i=0}^{k-1} N_i \equiv \varepsilon^{k(2l-1)} \begin{pmatrix} 1 & 0 \\ \sum_{i=0}^{k-1} b_{n-2i} & 1 \end{pmatrix} \pmod{u_{n-2k-1}}. \tag{4.33}$$

However by (4.13) we see that, for $i = 0, 1, \cdots, k$,

$$b_{n-2i} = (u_1 - \varepsilon) C_n \left(\prod_{h=i+1}^{k} u_{n-2h} \right)^2 \frac{u_{n-2i}^l - \varepsilon^l}{u_{n-2i} - \varepsilon}, \tag{4.34}$$

where $C_n = \prod_{h \geq k+1} u_{n-2h}^2$ satisfies by Lemma 4 (*i*)

$$(C_n, u_{n-2k-1}) = 1. \tag{4.35}$$

By using Lemma 4 (*ii*) and (*iii*), we see from (4.34) that

$$b_{n-2i} \equiv l\varepsilon^l (u_1 - \varepsilon) C_n \varepsilon^{2(k-i)} \pmod{u_{n-2k-1}} \tag{4.36}$$

for $i = 0, 1, \cdots, k$. Now we denote

$$D_k = \sum_{h=1}^{k} \varepsilon^{2h}, \tag{4.37}$$

and we obtain from (4.33)

$$\prod_{i=0}^{k-1} N_i \equiv \varepsilon^{k(2l-1)} \begin{pmatrix} 1 & 0 \\ l\varepsilon^l (u_1 - \varepsilon) C_n D_k & 1 \end{pmatrix} \pmod{u_{n-2k-1}}. \tag{4.38}$$

Finally, from (4.25) and (4.28) we have

$$\begin{pmatrix} A_{n-1} \\ A_n \end{pmatrix} = \left(\prod_{i=0}^{k-1} N_i \right) \begin{pmatrix} 0 & 1 \\ a_{n-2k} & b_{n-2k} \end{pmatrix} \begin{pmatrix} A_{n-2k-2} \\ A_{n-2k-1} \end{pmatrix}.$$

As $a_{n-2k} \equiv 0 \pmod{u_{n-2k-1}}$ by (4.12) since $n - 2k \geq 2$, this yields by using (4.36) and (4.38)

$$A_n \equiv l (u_1 - \varepsilon) C_n \varepsilon^{k(2l-1)+l} (D_k + 1) A_{n-2k-1} \pmod{u_{n-2k-1}}, \tag{4.39}$$

and the same holds for B_n. We now proceed as in the first case. As $d_{n-2k-1} \,|u_{n-2k-1}$ and $d_{n-2k-1} \,|(A_n, B_n)$, we see by (4.39) and (4.20) that

$$d_{n-2k-1} \left| l (u_1 - \varepsilon) C_n \varepsilon^{k(2l-1)+(n+1)l} (D_k + 1) (u_0 u_1 \cdots u_{n-2k-2})^{l-1} \right. . \tag{4.40}$$

As $d_{n-2k-1} \,|u_{n-2k-1}$, we have by Lemma 4 (*i*) and (4.35)

$$d_{n-2k-1} \,|l (u_1 - \varepsilon) (D_k + 1) ,$$

and therefore by (4.37)

$$d_{n-2k-1} \leq l (u_1 - \varepsilon) (k+1) \varepsilon^{2k}, \quad 0 \leq k \leq \frac{1}{2} (n-2). \tag{4.41}$$

By using (4.31) and (4.41), we see that

$$d_1 d_2 \cdots d_{n-1} \leq l^n (u_1 - \varepsilon)^n n! \varepsilon^{2n^2},$$

which proves (4.23) and (4.19) and completes the proof of Theorem 1.

5. An alternative proof in a special case

In this section, we give an alternative proof of Theorem 1 in the case where $l = 1$. Starting with (1.11), we can write

$$\gamma_{1,\varepsilon} - \sum_{k=0}^{n} \frac{\varepsilon^{2k}}{u_{2k}} = \sum_{k=n+1}^{\infty} \frac{\varepsilon^{2k}}{u_{2k}}. \tag{5.1}$$

We define for every integer $n \geq 0$

$$q_n = u_0 u_2 \cdots u_{2n}, \quad R_n = \sum_{k=n+1}^{\infty} \frac{\varepsilon^{2k}}{u_{2k}}. \tag{5.2}$$

We also define p_n by induction by $p_0 = 1$ and

$$p_{n+1} = p_n u_{2n+2} + \varepsilon^{2n+2} q_n. \tag{5.3}$$

With these notations, (5.1) becomes

$$\gamma_{1,\varepsilon} - \frac{p_n}{q_n} = R_n \tag{5.4}$$

and an easy induction using (5.3) and Lemma 4 (i) shows that $(p_n, q_n) = 1$ for every integer $n \geq 0$. Moreover, by Lemma 2, there exists a constant $\lambda > 0$ such that

$$\log u_{2n} = \lambda 4^n + O(1), \tag{5.5}$$

and consequently

$$\log q_n = \frac{\lambda}{3} 4^{n+1} + O(n). \tag{5.6}$$

Now let h be any given positive number, arbitrarily small. By (5.5) and (5.6) we can find a positive integer N such that for every $n \geq N$

$$\begin{cases} (\lambda - h)\, 4^n \leq \log u_{2n} \leq (\lambda + h)\, 4^n, \\ \frac{1}{3}(\lambda - h)\, 4^{n+1} \leq \log q_n \leq \frac{1}{3}(\lambda + h)\, 4^{n+1}, \\ -\log u_{2n+2} \leq \log R_n \leq -(1-h)\log u_{2n+2}. \end{cases} \tag{5.7}$$

Hence for $n \geq N$ we have

$$-3\frac{\lambda + h}{\lambda - h} \log q_n \leq \log R_n \leq -3\frac{(1-h)(\lambda - h)}{\lambda + h} \log q_n.$$

Therefore we obtain from (5.4)

$$\frac{1}{q_n^{\alpha}} \leq \gamma_{1,\varepsilon} - \frac{p_n}{q_n} \leq \frac{1}{q_n^{\beta}} \quad (n \geq N)\,, \tag{5.8}$$

where α and β are given by

$$\alpha = 3\frac{\lambda + h}{\lambda - h}, \quad \beta = 3\frac{(1-h)(\lambda - h)}{\lambda + h}. \tag{5.9}$$

By (5.8) we see at once that $\mu(\gamma_{1,\varepsilon}) \geq \beta$. As h is arbitrarily small, this proves that $\mu(\gamma_{1,\varepsilon}) \geq 3$ by letting $h \to 0$. Now we proceed to the proof that $\mu(\gamma_{1,\varepsilon}) \leq 3$. For this, we consider an irreducible rational number p/q and distinguish two cases.

First case: There exists $n \geq N$ such that $pq_n - qp_n = 0$. As the fractions p/q and p_n/q_n are both irreducible, we have $p = p_n$ and $q = q_n$ and from the left inequality in (5.8) we get

$$\left|\gamma_{1,\varepsilon} - \frac{p}{q}\right| \geq \frac{1}{q^\alpha}. \tag{5.10}$$

Second case: We have $pq_n - qp_n \neq 0$ for every $n \geq N$. In this case, fix a positive integer Q such that

$$\frac{Q}{q_N^{\beta-1}} \geq \frac{1}{2}$$

and assume that $|q| \geq Q$. Let $n \geq N$ be the least integer such that

$$\frac{|q|}{q_n^{\beta-1}} < \frac{1}{2}. \tag{5.11}$$

Then $n \geq N+1$ and by definition of n we can write

$$\log q_{n-1} \leq \frac{1}{\beta-1} \log (2|q|),$$

which yieds by using (5.7)

$$\log q_n \leq \frac{4}{3}(\lambda+h) 4^n \leq 4\frac{\lambda+h}{\lambda-h} \log q_{n-1} \leq \delta \log(2|q|), \tag{5.12}$$

where

$$\delta = 4\frac{\lambda+h}{(\lambda-h)(\beta-1)}. \tag{5.13}$$

As $pq_n - qp_n \neq 0$, we can write for $|q| \geq Q$

$$\begin{aligned} 1 &\leq |pq_n - qp_n| \leq |q| |q_n\gamma_{1,\varepsilon} - p_n| + q_n |q\gamma_{1,\varepsilon} - p| \\ &\leq \frac{|q|}{q_n^{\beta-1}} + |q_n| |q\gamma_{1,\varepsilon} - p| < \frac{1}{2} + q_n |q\gamma_{1,\varepsilon} - p|, \end{aligned}$$

where we have used (5.8) and (5.11). This implies

$$\left|\gamma_{1,\varepsilon} - \frac{p}{q}\right| > \frac{1}{2|q|q_n},$$

and by using (5.12)

$$\left|\gamma_{1,\varepsilon} - \frac{p}{q}\right| > \frac{1}{(2|q|)^{\delta+1}} > \frac{1}{|q|^{\delta+1+h}} \tag{5.14}$$

for $|q| \geq Q' \geq Q$. By considering (5.10) and (5.14) we see that

$$\mu(\gamma_{1,\varepsilon}) \leq \max(\alpha, \delta+1+h).$$

Letting $h \to 0$ yields $\mu(\gamma_{1,\varepsilon}) \leq 3$, which completes the proof.

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