Math 456 Lecture Notes:
Bessel Functions and their Applications to Solutions of Partial Differential Equations

Vladimir Zakharov

June 3, 2009

1 Gamma Function

Gamma function $\Gamma(s)$ is defined as follows:

$$\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} dt \tag{1}$$

As far as:

$$t^{s-1} = \frac{1}{s} \frac{\partial}{\partial t} t^s \tag{2}$$

By plugging (2) into (1) we get

$$s\Gamma(s) = \int_0^\infty e^{-t} \frac{d}{dt} t^s dt = e^{-t} t^s|_0^\infty + \int_0^\infty e^{-t} t^s dt \tag{3}$$

or

$$s\Gamma(s) = \Gamma(s+1) \tag{4}$$

Then $\Gamma(1) = 1$ and $\Gamma(2) = 1$.
By induction we obtain:

$$\Gamma(n+1) = n! \tag{5}$$

Then

$$\begin{aligned}
\Gamma(s) &= \frac{1}{s}\Gamma(s+1) \\
\Gamma(s) &\to \frac{1}{s} \; if \;\; s \to 0 \\
\Gamma(s-1) &= \frac{1}{s(s-1)}\Gamma(s+1) \\
\Gamma(s-n) &= \frac{1}{(s-n)(s-n+1)\cdots s}\Gamma(s+1)
\end{aligned} \tag{6}$$

$\Gamma(s)$ has holes at all negative integral values of s.

To find asymptotic behavior of Gamma-function as $s \to \infty$, we use so called "Laplace Method."

$$\Gamma(s+1) = \int_0^\infty e^{-t} t^s dt = \int_0^\infty e^{-\phi(t,s)} dt \tag{7}$$

$$\phi(t,s) = t - s\ln(t)$$

Function $\phi(t,s)$ has a minimum at $t = s$. Indeed:

$$\frac{\partial \phi}{\partial t} = 1 - \frac{s}{t} = 0 \;\; if \;\; t = s \tag{8}$$

Near this minimum:

$$\begin{aligned} \phi \quad &= \phi_o(s) + \frac{1}{2}\phi''(s)\tau^2 + \cdots, \quad \tau = t - s \\ \phi_o(s) \quad &= s - s\ln(s) \\ \phi''(s) \quad &= \frac{1}{s} \end{aligned}$$

Now we will replace in (7) $\phi(t,s)$ to its approximate value (8) and go from integration by t to integration by τ. Whithout loss of accuracy we can consider that $-\infty < \tau < \infty$.

Then

$$\begin{aligned} \Gamma(s+1) \quad &\approx e^{-\phi_o(s)} \int_{-\infty}^{\infty} e^{-\frac{\tau^2}{2s}} d\tau \\ e^{-\phi_o(s)} \quad &= \left(\frac{s}{l}\right)^s \end{aligned} \tag{9}$$

Now we replace $\tau = \sqrt{2s}y$ and remember that $\int_{-\infty}^{\infty} e^{-y^2} dy = \sqrt{\pi}$. We end up with the following answer:

$$\begin{aligned} \Gamma(s+1) \quad &\approx \sqrt{2\pi s}\left(\frac{s}{l}\right)^n \\ n! \quad &\approx \sqrt{2\pi n}\left(\frac{n}{l}\right)^n \end{aligned} \tag{10}$$

This is the Stirling approximation for $n = 5$, where $n! = 120$. The Stirling approximation gives $5! \approx 118.045$. The accuracy of the Stirling approximation is reasonable. We accept without proof:

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)} \tag{11}$$

where $\Gamma^2(\frac{1}{2}) = \pi$ so $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

2 Bessel Equation Appears

Let us try to solve the diffusion equation

$$u_t = \chi \Delta u \tag{12}$$

inside the disk of radius a in polar coordinates:

$$\Delta u = \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} \tag{13}$$

We impose boundary conditions $u(r = a) = 0$ with initial data $u(t = 0) = \phi(r, \theta)$.
In polar coordinates the previous equation becomes:

$$u_t = \chi \Big(\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} \Big) \tag{14}$$

Partial solutions to this equation can be found of the following form:

$$u(r, \theta, t) = e^{in\theta} e^{-\chi t k^2} R(r) \tag{15}$$

The radial part $R(r)$ satisfies the equation

$$\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial R}{\partial r} + \Big(k^2 - \frac{n^2}{r^2} \Big) R = 0 \tag{16}$$

k^2 can take discrete values $k^2 = k_1^2, \cdots, k_N^2, \cdots$
Corresponding radial functions $R_N(r)$ satisfy the Dirichlet condition $R_N(a) = 0$.
By change of variables $z = kr$ we have:

$$\frac{1}{z}\frac{\partial}{\partial z}z\frac{\partial R}{\partial z}+\left(1-\frac{s^2}{z^2}\right)R=0 \tag{17}$$

where we have replaced $n^2 = s^2$, assuming that s is an arbitrary real number.
The previous equation is the Bessel equation. At $z \to 0$ it becomes the equipotent equation:

$$\frac{1}{z}\frac{\partial}{\partial z}z\frac{\partial R}{\partial z}-\frac{s^2}{z^2}R=0 \tag{18}$$

which can be solved explicitly:

$$R = C_1 z^s + C_2 z^{-s} \tag{19}$$

One can seek a solution of (17) in the form

$$R = \left(\frac{z}{2}\right)^s F(z,s) \tag{20}$$

F satisfies the equation:

$$F'' + \frac{2s+1}{z}F' + F = 0 \tag{21}$$

The solution of equation (21) can be found in the form of series:

$$F = \sum_{k=0}^{\infty} C_k \left(\frac{z}{2}\right)^{2k} \tag{22}$$

After differentiating by z, the first term in (22) vanishes. One can see that:

$$F'' + \frac{2s+1}{z}F' = \sum_{k=1}^{\infty} C_k \frac{[(2k-1)2k + 2k(2s+1)]}{4}\left(\frac{z}{2}\right)^{2(k-1)} \tag{23}$$

Let us replace $k \to k+1$. Now:

$$F'' + \frac{2s+1}{z}F' = \sum_{k=0}^{\infty} C_{k+1}(k+1)(k+s+1)\left(\frac{z}{2}\right)^{2k} \tag{24}$$

By substitution we will finally get:

$$C_{k+1}(k+1)(k+s+1)+C_k=0 \tag{25}$$

or

$$C_{k+1}=-\frac{C_k}{(k+1)(k+s+1)} \tag{26}$$

This can then be solved as follows:

$$C_k=\frac{(-1)^k}{\Gamma(k+1)\Gamma(k+s+1)}=\frac{(-1)^k}{k!\Gamma(k+s+1)} \tag{27}$$

In particular for integral s = n:

$$C_k=\frac{(-1)^k}{k!(n+k)!} \tag{28}$$

3 Bessel Function

The Bessel function $J_s(z)$ is defined by the series:

$$J_s(z)=\left(\frac{z}{2}\right)^s\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\Gamma(s+k+1)}\left(\frac{z}{2}\right)^{2k} \tag{29}$$

This series converges for all z on the complex plane, thus $J_s(z)$ is the entire function. If $z\longrightarrow 0$, then

$$J_s(z)\rightarrow\left(\frac{z}{2}\right)^s\frac{1}{\Gamma(s+1)} \tag{30}$$

If s^2 is not an integer, then $J_{-s}(z)$ is the second solution of the Bessel equation. Now:

$$J_{-s}(z)\rightarrow\left(\frac{z}{2}\right)^{-s}\frac{1}{\Gamma(-s+1)} \tag{31}$$

$J_s(z)$ is regular at $z\longrightarrow 0$, while $J_{-s}(z)$ is singular. So, let $s\rightarrow\infty$. By the use of the Stirling Formula we get:

$$J_s(z)\rightarrow\frac{1}{\sqrt{2\pi s}}\left(\frac{lz}{2s}\right)^s \tag{32}$$

What happens if $s \to -n$? Notice that $(-s+1)\Gamma(-s+1) = \Gamma(-s+2)$. Hence (31) can be re-written as follows:

$$J_{-s}(z) \to \left(\frac{z}{2}\right)^{-s} \frac{-s+1}{\Gamma(-s+2)} \tag{33}$$

Let $s \to 1$, then (33) tends to zero and the first term in (29) vanishes. All the other terms are finite thus:

$$J_{-1}(z) = \left(\frac{z}{2}\right)^{-1} \sum_{k=1}^{\infty} \frac{(-1)^k}{k!\Gamma(k)} \left(\frac{z}{2}\right)^{2k} \tag{34}$$

By replacing $k \to k+1$, we get

$$J_{-1}(z) = -\left(\frac{z}{2}\right) \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)!\Gamma(k+1)} \left(\frac{z}{2}\right)^{2k} \tag{35}$$

Now $(k+1)! = (k+1)k!$ and $(k+1)\Gamma(k+1) = \Gamma(k+2)$, hence:

$$J_{-1}(z) = -\frac{z}{2} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k+2)} \left(\frac{z}{2}\right)^{2k} \tag{36}$$

On the right hand side of this equation we have $-J_1(z)$, hence

$$J_{-1}(z) = -J_1(z) \tag{37}$$

In a similar way, we obtain:

$$J_{-n}(z) = (-1)^n J_n(z) \tag{38}$$

For integral n, J_n and J_{-n} are linearly dependent, and we must construct a second solution of the Bessel equation by another way. Let us calculate the following derivative:

$$\begin{aligned}
\frac{d}{dz} z^{-s} J_s(z) &= \frac{1}{2^s} \frac{d}{dz} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(s+k+1)} \left(\frac{z}{2}\right)^{2k} \\
&= \frac{1}{2^s} \sum_{k=1}^{\infty} \frac{(-1)^k}{k!\Gamma(s+k+1)} 2k \left(\frac{z}{4}\right) \left(\frac{z}{2}\right)^{2(k-1)} \\
&= -\frac{z}{2^{s+1}} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(s+k+2)} \left(\frac{z}{2}\right)^{2k}
\end{aligned}$$

By multiplying by z^s we get:

$$z^s \frac{d}{dz} z^{-s} J_s(z) = -J_{s+1}(z) \tag{39}$$

or

$$J_{s+1}(z) = \frac{s}{z} J_s(z) - J_s'(z) \tag{40}$$

Let us consider now the following function:

$$z^s J_s = 2^s \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(s+k+1)} \left(\frac{z}{2}\right)^{2(s+k)} \tag{41}$$

After differentiating by z and multpling by z^{-s}, we obtain:

$$\begin{aligned} z^{-s} \frac{d}{dz} (z^s J_s) \qquad &= \left(\frac{z}{2}\right)^{s-1} \sum_{k=0}^{\infty} \frac{(-1)^k (s+k)}{k!\Gamma(s+k+1)} \left(\frac{z}{2}\right)^{2k} \\ &= \left(\frac{z}{2}\right)^{s-1} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(s+k)} \left(\frac{z}{2}\right)^{2k} \\ &= J_{s-1} \end{aligned}$$

Finally,

$$z^{-s} \frac{d}{dz} (z^s J_s(z)) = J_{s-1}(z) \tag{42}$$

$$J_{s-1}(z) = \frac{s}{z} J_s + J_s'(z) \tag{43}$$

By combining (40) and (43) we get

$$J_{s+1}(z) + J_{s-1}(z) = \frac{2s}{z} J_s(z) \tag{44}$$

4 Bessel Functions of Half-integral Index

Let us introduce the function g defined as follows:

$$J_s = \frac{1}{z^{\frac{1}{2}}} g = \sqrt{\frac{1}{z}} g \tag{45}$$

After plugging into the Bessel equation (17), one realizes that g satisfies the equation:

$$g'' + \left(1 - \frac{s^2 - \frac{1}{4}}{z^2}\right) g = 0 \tag{46}$$

Let $s = \frac{1}{z}$. Then,

$$g'' + g = 0 \tag{47}$$

as far as J_s is regular at z $\to \infty$

$$g = c\sin(z) \qquad\qquad J_s \to cz^{\frac{1}{2}}$$

To find c, we remember the asymptotics of the Bessel functions at $z \to 0$.

$$J_{\frac{1}{2}}(z) \to \left(\frac{z}{2}\right)^{\frac{1}{2}} \frac{1}{\Gamma(\frac{3}{2})}$$

$$\Gamma\left(\frac{3}{2}\right) = \Gamma\left(\frac{1}{2} + 1\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2}$$

$$J_{\frac{1}{2}}(z) \to \sqrt{\frac{2z}{\pi}}$$

Finally,

$$J_{\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}} \sin(z) \tag{48}$$

According to (40), all $J_{n+\frac{1}{2}}(z)$ are expressed through a combination of power and trigonometric functions. In particular,

$$J_{\frac{3}{2}}(z) = -z^{\frac{1}{2}} \frac{d}{dz}(z^{\frac{-1}{2}} J_{\frac{1}{2}}(z)) = -\sqrt{\frac{2}{\pi}} z^{\frac{1}{2}} \frac{d}{dz} \frac{\sin(z)}{z} = \sqrt{\frac{2}{\pi}} \left(\frac{1}{z^{\frac{3}{2}}} \sin(z) - \frac{1}{z^{\frac{1}{2}}} \cos(z)\right)$$

or

$$J_{\frac{3}{2}}(z) = \sqrt{\frac{2}{\pi z}}\Big(\frac{\sin(z)}{z} - \cos(z)\Big)$$

Separating this procedure we get:

$$J_{\frac{5}{2}}(z) = \sqrt{\frac{2}{\pi z}}\Big[\Big(\frac{3}{z^2} - 1\Big)\sin(z) - \frac{3}{z}\cos(z)\Big]$$

Now, let $z \to \infty$ in equation (40). One can put approximately:

$$J_{s+1}(z) \approx -J_s^{'}(z)$$

Then, we have:

$$\begin{aligned} J_{\frac{3}{2}} &\to -\sqrt{\frac{2}{\pi z}}\cos(z) \quad J_{\frac{5}{2}} \to -\sqrt{\frac{2}{\pi}}\sin(z) \\ J_{\frac{7}{2}} &\to \quad \sqrt{\frac{2}{\pi z}}\cos(z) \quad \cdots \end{aligned}$$

One can see that $J_{n+\frac{1}{2}}(z)$ has an infinite amount of zeros on the real axis. The same statement is correct for all Bessel functions.

5 Integral Representation

Let us study the integral:

$$A_n(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{iz\sin(\theta) - in\theta} d\theta \tag{49}$$

To evaluate this integral, we use the Taylor expansion of the exponent:

$$e^{iz\sin(\theta)} = \sum_{p=0}^{\infty}\frac{1}{p!}(iz\sin(\theta))^p = \sum_{p=0}^{\infty}\frac{1}{p!}\Big(\frac{z}{2}\Big)^p(e^{i\theta} - e^{-i\theta})^p \tag{50}$$

Now, notice that the integral:

$$I_{p,\pi} = \frac{1}{2\pi}\int_{-\pi}^{\pi}(e^{i\theta} - e^{-i\theta})^p e^{-in\theta} d\theta = 0 \quad if \quad p < 0 \tag{51}$$

Then, we denote $p = n + q$. The integrand in (51) can be presented in the form:

$$\frac{1}{2\pi}(e^{i\theta} - e^{-i\theta})^{n+q} e^{-in\theta} = (1 - e^{-2i\theta})^n (e^{i\theta} - e^{-i\theta})^q$$

Suppose that q is odd ($q = 2k + 1$). All terms in the first parentheses are even powers of $e^{-i\theta}$, while all terms in the second parentheses are odd powers (positive or negative) on $e^{-i\theta}$. As a result, the integrand is a linear combination of odd powers of $e^{-i\theta}$. Thus the integral is zero, and we can put $q = 2k$. We obtain the following intermediate result:

$$A_n(z) = \left(\frac{z}{2}\right)^n \sum_{n=0}^{\infty} \frac{1}{(n+2k)!}\left(\frac{z}{2}\right)^k I_{k,n} \tag{52}$$

Where

$$I_{k,n} = \frac{1}{2\pi}\int_{-\pi}^{\pi}(e^{i\theta} - e^{-i\theta})^{n+2k} e^{-in\theta} d\theta \tag{53}$$

To calculate $I_{k,n}$, we use the binomial expansion in the parentheses. In this expansion, we are interested only in the single term proportional to $e^{in\theta}$. All other terms after multiplication to (53) and integration over θ are cancelled. Hence,

$$(e^{i\theta} - e^{-i\theta})^{n+2k} \approx \frac{(n+2k)!}{k!(n+k)!}(e^{i\theta})^{n+k}(-e^{-i\theta})^k = \frac{(-1)^k(n+2k)!}{k!(n+k)!}e^{in\theta}$$

and

$$I_{n,k} = \frac{(-1)^k(n+2k)!}{k!(n+k)!} \tag{54}$$

By plugging (54) into (52), we get finally:

$$A_n(z) = \left(\frac{z}{2}\right)^n \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(n+k)!}\left(\frac{z}{2}\right)^k = J_n(z)$$

We obtained the integral representation for $J_n(z)$:

$$J_n(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{iz\sin(\theta)-in\theta}d\theta \tag{55}$$

This result is correct for positive n. Let us notice that,

$$J_n(-z) = (-1)^n J_n(z) \tag{56}$$

Bessel functions of even order are even functions on z, while functions of odd order are odd. Now, we can find $A_n(z)$ at negative n. Let us change simultaneously the signs on z and n.

$$A_{-n}(-z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-iz\sin(\theta)+in\theta}d\theta$$

Now by replacing $\theta \to -\theta$, we restore the previous result. Hence,

$$\begin{aligned} A_{-n}(-z) &= A_n(z) = J_n(z) \\ A_{-n}(z) &= J_n(-z) = (-1)^n J_n(z) \end{aligned} \tag{57}$$

Finally, for all integrals on

$$A_n(z) = (-1)^n J_n(z)$$

notice also that J_n is real. Then (55) can be rewritten as follows:

$$J_n(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos(z\sin\theta - n\theta)d\theta \tag{58}$$

Now, look at $e^{iz\sin(\theta)}$. This is a periodic function which can be expanded in the Fourier series. Apparently,

$$e^{iz\sin(\theta)} = \sum_{n=-\infty}^{\infty} J_n(z)e^{in\theta} = J_0(z) + \sum_{n=1}^{\infty} J_n(z)(e^{in\theta} + (-1)^n e^{-in\theta}) \tag{59}$$

The separating of imaginary and real parts in (59) gives us:

$$\cos(z\sin(\theta)) = J_0(z) + 2\sum_{k=1}^{\infty} J_{2k}(z)\cos(2k\theta) \tag{60}$$

$$\sin(z\sin(\theta)) = 2\sum_{k=-\infty}^{\infty} J_{2k+1}(z)\sin((2k+1)\theta)$$

By introducing $t = e^{i\theta}$, one can transform (59) to the following expansion:

$$e^{\frac{z}{2}(t-\frac{1}{t})} = \sum_{n=-\infty}^{\infty} J_n(z)t^n \tag{61}$$

This means that $F(z,t) = e^{\frac{z}{2}(t-\frac{1}{t})}$ is a 'generating function' for the entire community of Bessel functions of integral orders.

6 Asymptotic behavior at $z \to \infty$

To find the asymptotic behavior of the Bessel functions at $z \to \infty$, we will use the device similar to the one used for the derivation of the Stirling formula. We present integral (55) in the form:

$$J_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i\Phi(z,\theta)}d\theta \tag{62}$$

$$\Phi(z,\theta) = z\sin(\theta) - n\theta \tag{63}$$

If $z \to \infty$, the integrand is the fast oscillation function everywhere except the two points where $\frac{d\Phi}{d\theta} = 0$. These points are defined by the equation:

$$\begin{aligned} z\cos(\theta) &\quad = n \\ at &\quad z \to \infty \\ \cos(\theta) \to 0 &\quad \theta \to \pm\frac{\pi}{2} \end{aligned}$$

The contributions of points $\theta^{\pm} = \pm\frac{\pi}{2}$ give complex conjugated results. Hence, it is enough to study the neighbourhood of the point $\theta = \frac{\pi}{2}$. Let us introduce $\theta = \frac{\pi}{2} + \tau$. For small τ,

$$\Phi(z,\theta) \approx z - \frac{n\pi}{2} - \frac{1}{2}z\tau^2 \tag{64}$$

Integral (62) can be replaced approximately by the following integral:

$$J_n(z) = \frac{1}{\pi}\Re e^{i(z-\frac{\pi n}{2}-\frac{\pi}{4})}\int_{\infty}^{\infty} e^{\frac{-iz}{2}\tau^2}d\tau, \quad \Re \equiv Real\ part$$

Let us make the change of varibles:

$$\tau = \sqrt{\frac{2}{iz}}y, \quad \frac{1}{\sqrt{i}} = e^{\frac{-\pi i}{4}}$$

Then,

$$J_n(z) = \frac{\sqrt{2}}{\pi\sqrt{z}} \Re e^{i(z-\frac{\pi n}{2}-\frac{\pi}{4})} \int_{\frac{-i\pi}{4}*\infty}^{\frac{i\pi}{4}*\infty} e^{-y^2} dy \tag{65}$$

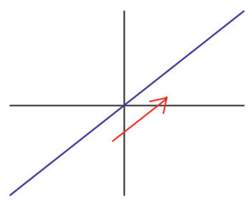


Figure 1: Contour of Integration.

Integration is going in the complex plane along the straight line turned by 45° with respect to the real axis. This is demonstrated in Fig. 1.

However, the contour of integration can be turned back and returned to the real axis (To justify this fact, we need to use some elemetns of complex analysis. But, this is true.) In other words, the integral in (65) can be replaced by the integral $\int_{-\infty}^{\infty} e^{-y^2 dy} = \sqrt{\pi}$. We end up with the following result:

$$J_n(z) \to \sqrt{\frac{2}{\pi z}} \cos\left(z - \frac{\pi n}{2} - \frac{\pi}{4}\right) \tag{66}$$

We derived this expression only for integral n. In fact, this is correct for all s. To prove this, we have to use a more sophisticated integral respresenation for $J_s(z)$ which is valid not only for integrals. In general,

$$J_s(z) \to \sqrt{\frac{2}{\pi z}} \cos\left(z - \frac{\pi s}{2} - \frac{\pi}{4}\right) \tag{67}$$

In particular,

$$J_{\frac{1}{2}} \to \sqrt{\frac{2}{\pi z}} \cos\left(z - \frac{\pi}{2}\right) \to \sqrt{\frac{2}{\pi z}} \sin(z)$$

This is the unique Bessel function coinciding with its own asymptotic behavior.

7 Zeros of Bessel function

It is clear from (67) that the Bessel function $J_s(z)$ has an infinite amount of zeros for the half axis $0 < z < \infty$. Let us denote these zeros as a_N^s, where $N = 1, 2, ...\infty$. From (67), one can conclude that the distance between two neighboring zeros tends to π.

$$a_{N+1}^s - a_N^s \to \pi \quad as\ N \to \infty \tag{68}$$

The first five Bessel functions of integral order are plotted on Figure 1. The first five of each are presented in Table 1. Notice that:

$$a_5^5 - a_4^5 = 3.2377$$

While:

$$a_5^0 - a_4^0 = 3.1394$$

Both values are close to π. The derivatives of Bessel functions have the following asymptotic behavior:

$$J_s'(z) \to -\sqrt{\frac{2}{z\pi}} \sin\left(z - \frac{s\pi}{2} - \frac{\pi}{4}\right) \tag{69}$$

The derivatives of $J_s'(z)$ also have an infinite amount of zeros b_N^s. Again:

$$b_{N+1}^s - b_N^s \to \pi\ if\ N \to \infty \tag{70}$$

Table 1: ROOTS of the FUNCTION $J_n(x)$ are given in the following table.

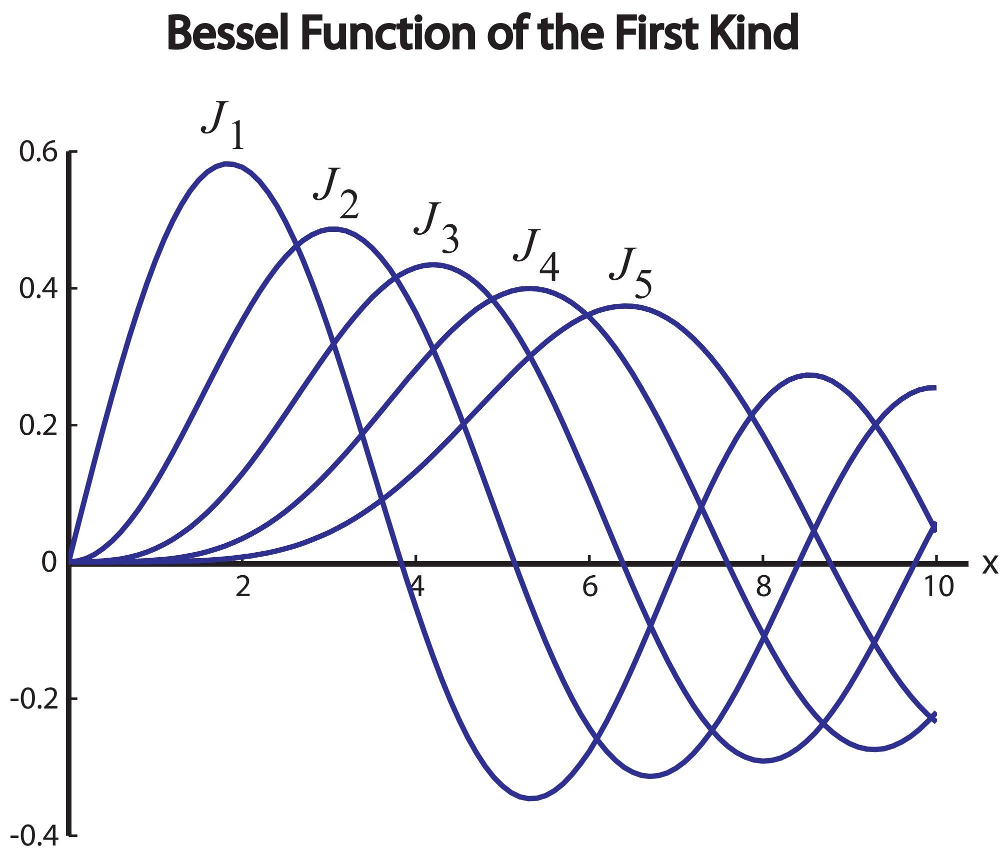
zero	$J_0(x)$	$J_1(x)$	$J_2(x)$	$J_3(x)$	$J_4(x)$	$J_5(x)$
1	2.4048	3.8317	5.1336	6.3802	7.5883	8.7715
2	5.5201	7.0156	8.4172	9.7610	11.0647	12.3386
3	8.6537	10.1735	11.6198	13.0152	14.3725	15.7002
4	11.7915	13.3237	14.7960	16.2235	17.6160	18.9801
5	14.9309	16.4706	17.9598	19.4094	20.8269	22.2178

Table 2: The ROOTS of its DERIVATIVES are given in the following table.

zero	$J'_0(x)$	$J'_1(x)$	$J'_2(x)$	$J'_3(x)$	$J'_4(x)$	$J'_5(x)$
1	3.8317	1.8412	3.0542	4.2012	5.3175	6.4156
2	7.0156	5.3314	6.7061	8.0152	9.2824	10.5199
3	10.1735	8.5363	9.9695	11.3459	12.6819	13.9872
4	13.3237	11.7060	13.1704	14.5858	15.9641	17.3128
5	16.4706	14.8636	16.3475	17.7887	19.1960	20.5755

Zeros of the first five $J'_n(z)$ are represented in Table 2.

Figure 1


Bessel Function of the First Kind
J_1
J_2
J_3
J_4
J_5
0.6
0.4
0.2
0
-0.2
-0.4
2
4
6
8
10
x

Let us rewrite the Bessel equation as follows:

$$\frac{d}{dz}zJ_s' + zJ_s - \frac{s^2}{z}J_s = 0 \tag{71}$$

By multiplying by $2zJ'$, we get

$$\frac{d}{dz}(z^2J_s'^2 - s^2J_s^2) + 2z^2JJ' = 0$$

$$2z^2JJ' = z^2\frac{d}{dz}J^2 = \frac{d}{dz}z^2J^2 - 2zJ^2$$

Finally

$$2zJ_s^2 = \frac{d}{dz}\Big[z^2J_s'^2 + (z^2 - s^2)J_s^2\Big] \tag{72}$$

Integrating (72) with respect to z from 0 to a_N, we get:

$$\int_0^{a_N} zJ_s^2(z)dz = \frac{1}{2}a_N^2J_s'^2(a_N) = \frac{1}{2}a_N^2J_{s\pm1}^2(a_N) \tag{73}$$

The last part of equation (72) follows from equations (43) and (44). In virtue of (43), $J_s'(a_N) = J_{s-1}(a_N)$. In virtue of (44), $J_{s+1}(a_N) = -J_{s-1}(a_N)$, thus:

$$J_{s+1}^2(a_N) = J_{s-1}^2(a_N) = J_s'^2(a_N) \tag{74}$$

From Table 1, one can see that the first zero a_0^n grows with n. The following statement is correct: The number of zeros of $J_s(z)$ on the interval

$$0 < z < \left(m + \frac{s}{z} + \frac{1}{4}\right)\pi \tag{75}$$

is exactly m. Putting $m = 1$ into (75) we get:

$$a_1^s < \left(\frac{3}{4} + \frac{s}{2}\right)\pi \tag{76}$$

For $s = 5$, we get $a_1^5 < 10.35$. In reality $a_1^5 = 8.7715$. We see that this estimate is rather accurate.

8 Orthogonality and Fourier-Bessel series

Let $J_s(z)$ be the Bessel function of order (or index) s. Let a_N^s be its zeros such that $J_s(a_N^s) = 0$. Suppose that $0 < r < a$ is an interval on the real azis. We consider now the set of the function $R_N^{(s)}(r) = J_s(\frac{r}{R}a_N^s)$. This is the set of functions against the weight r. In other words

$$\int_0^a R_N^{(s)}(r)R_M^{(s)}(r)\, rdr = 0 \;\; if \;\; N \neq M \tag{77}$$

To prove this fact, we first mention that

$$R_N^s(r) = J_s(a_N^s) = 0 \tag{78}$$

Then it is easy to check that these functions satisfy the equations

$$\frac{1}{r}\,\frac{\partial}{\partial r}\,r\,\frac{\partial R_N^s}{\partial r} + \left(k_N - \frac{s^2}{r^2}\right) R_N^s = 0, \qquad k_N = \frac{a_N}{a} \tag{79}$$

$$\frac{1}{r}\,\frac{\partial}{\partial r}\,r\,\frac{\partial R_M^s}{\partial r} + \left(k_M - \frac{s^2}{r^2}\right) R_M^s = 0. \qquad k_M = \frac{a_M}{a} \tag{80}$$

By multiplying these equations by rR_M and rR_N respectively and subtracting the results, we get

$$R_M^s\frac{\partial}{\partial r}r\frac{\partial R_N^s}{\partial r} - R_N^s\frac{\partial}{\partial r}r\frac{\partial R_M^s}{\partial r} = (k_M^2 - k_M^2)rR_NR_M \tag{81}$$

The left hand side can be rewritten as the following:

$$\frac{\partial}{\partial r}r[R_M, R_N] = (k_M^2 - k_M^2)rR_NR_M \tag{82}$$

$$[R_M, R_N] = R_M\frac{\partial R_M}{\partial r} - R_N\frac{\delta R_N}{\delta r} \tag{83}$$

$$[R_M, R_N]|_{r=a} = 0 \tag{84}$$

Then if $k_M^2 \neq k_M^2$, integration from zero to a leads to the condition of (77). Notice that we could replace functions $R_N^s(r)$ by $\widetilde{R}_n^s(r) = J_s(\frac{r}{a}b_N^s)$. They satisfy the condition $\widetilde{R}_n'(a) = 0$. In this case, equation (8.8) is again satisfied. The Wronskian $[R_M, R_N]$ is zero at $r = a$. Hence function $\widetilde{R}_N^s(r)$ satisfies the orthogonality condition (77).

Suppose that $f(r)$, $0 < r < a$, is some real or complex function defined on the interval $(0, r)$. We can represent this function as a linear combination of $R_N^s(r)$.

Let

$$f(r) = \sum_{N=1}^{\infty} f_N R_N^s(r) \tag{85}$$

By multiplying this to $rR_M^s(r)$ and integrating, we get

$$f_N = \frac{1}{\lambda_N^{s2}} \int_0^a f(r) r R_N^s(r) dr = \frac{1}{\lambda_N^{s2}} \int_0^a f(r) r J_N^s\Big(\frac{r}{a} a_N\Big) dr$$

Here

$$\lambda_N^2 = \int_0^a r R_N^5(r) dr = \frac{1}{2}\frac{a^2}{a_N^2} \int_0^{a_N} z J_z^2(z) dz = \frac{1}{2} a^2 J_{s\pm 1}^2(a_N) \tag{86}$$

Now one important remark, all functions $R_N^s(r) \to (\frac{r}{2a} a_N)^s$ at $r \to 0$. It means that the series (78) reasonably converges if the function $f(r)$ behaves at $r \to$ as

$$f(r) \to cr^s \tag{87}$$

If the asymptotics of (80) holds, the conditions for the convergence of the series are very similar to corresponding conditions for the standard Fourier series. In particular, if $f(a) = 0$ $|f'(r)| < C$, where C is some arbitrary constant, this series converges absolutely and uniformly on $0 < r < a$.

A function $f(r, \theta)$ defined in the disk $0 < r < a$ can be expanded in this disk in the Bessel - Fourier series. First, present $f(r, \theta)$ as a Fourier series in angles.

$$f(r, \theta) = \sum_{n=-\infty}^{\infty} f_n(r) e^{in\theta} \tag{88}$$

$$f_n(r) = \frac{1}{2\pi} \int_0^{2\pi} f(r, \theta) e^{-in\theta} d\theta \tag{89}$$

What is asymptotic of $f_n(r)$ if $r \to 0$? Let us return to the Cartesian coordinates $(x = r\cos\theta, y = r\sin\theta)$. Let $f_n(\theta)$ be presented as follows:

$$f_n = f_0(r) + r f_1(\theta) + \frac{1}{2} r^2 f_2(\theta) + \cdots + \frac{1}{r} R^{n-1} f_{n-1}(\theta) \tag{90}$$

$$f_1(\theta) = f_x \cos\theta + f_y \sin\theta$$

All other $f_n(\theta)$ are trigonometric polynomials of order $n - 1$. Apparently:

$$\int_0^{2\pi} f_k(\theta)e^{-in\theta}d\theta = 0 \quad if \quad k < n$$

Hence $f_n(r) \to P_n r^n$ as $r \to 0$, with P_n some constant, and functions $f_n(r)$ are god for expansion in series of Fourier function of order n.

Finally

$$f(r,\theta) = \sum_{n=-\infty}^{\infty}\sum_{N=1}^{\infty} f_{nN}e^{in\theta}J_n\left(\frac{ra_N}{a}\right) \tag{91}$$

$$f_{nN} = \frac{1}{2\pi\lambda_{nM}^2}\int_0^{2\pi} e^{-in\theta}d\theta\int_0^a rJ_n\left(\frac{ra_N}{a}\right)f(r,\theta)dr \tag{92}$$

In particular, if $f(x,y) = \delta(x-x_0)\delta(y-y_0) = r_0\delta(\theta-\theta_0)\delta(r-r_0)$,

$$f_{nN} = \frac{1}{2\pi\lambda_{nM}^2}r_0e^{-in\theta_0}J_N\left(\frac{r_0}{a}a_N\right) \tag{93}$$

Series (84) are especially good and fast converging if $f(r,\theta)$ satisfies the Dirichlet condition $f(a,\theta) = 0$. If this function satisfies the Neumann condition ($f_r(a,\theta) = 0$), one can use the better following set of orthogonal functions:

$$1,\ J_n\left(\frac{r}{a}b_1\right), \cdots\ J_n\left(\frac{r}{a}b_n\right), \cdots$$

9 Application of the Fourier - Bessel series to solutions of PDEs in circular domains

In this chapter, we apply Bessel function to solution of boundary problems for some basic equations of mathematical physics. We will solve equations inside the circle of radius a, $0 < r < a$, with zero boundary conditions on the circle.

First, we will start with the Poisson equation

$$\Delta U = \frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} = f(r,\theta) \tag{94}$$

$$U|_{r=a} = 0$$

We expand $f(r,\theta)$ in the Bessel - Fourier series

$$f(r,\theta) = \sum_{N=1}^{\infty} \sum_{n=-\infty}^{\infty} f_{nN} e^{in\theta} J_n\Big(\frac{ra_N^n}{a}\Big) \tag{95}$$

$$f_{nN} = \frac{1}{2\pi\lambda_{nM}^2} \int_0^{2\pi} e^{-in\theta} d\theta \int_0^a r J_n\Big(\frac{ra_N^n}{a}\Big) f(r,\theta) dr \tag{96}$$

A solution of the Poisson equation can be found in the form of a similar series

$$U(r,\theta) = \sum_{N=1}^{\infty} \sum_{n=-\infty}^{\infty} U_{nN} e^{in\theta} J_n\Big(\frac{ra_N^n}{a}\Big) \tag{97}$$

The coeffiecients U_{nN} and f_{nN} are connected by a simple relation

$$U_{nN} = -\frac{a^2}{(a_N^n)^2} f_{nN} \tag{98}$$

Suppose that $f(x,y) - \delta(x-x_0)\delta(y-y_0) = r_0\delta(r-r_0)\delta(\theta-\theta_0)$ and $x_0 = r_0\cos\theta_0$, $y_0 = r_0 \sin\theta_0$. Here we used the relation

$$dxdy = rdrd\theta \tag{99}$$

$$f_n N = \frac{1}{2\pi\lambda_{nN}^2} e^{-in\theta} r_0 J_n\Big(\frac{r_0}{a} a_N^n\Big) \tag{100}$$

Now,

$$f(r,\theta) = G(r,r_0,\theta-\theta_0) = \frac{r_0}{2\pi} \sum_{n=-\infty}^{\infty} \sum_{N=1}^{\infty} e^{in(\theta-\theta_0)} \frac{J_n(\frac{ra_N^n}{a})}{\lambda_{nN}^2} J_n\Big(\frac{r_0 a_N^n}{a}\Big)$$

$G(r,r_0,\theta-\theta_0)$ is the Green function for the Poisson equation. In other words:

$$U(r,\theta) = \int_0^{2\pi} d\theta_0 \int_0^a G(r,r_0,\theta-\theta_0) f(r_0,\theta_0) dr_0 \tag{101}$$

Now we will solve the diffusion equation

$$\frac{\partial u}{\partial r} = \kappa\Big(\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\Big)$$

inside of the circle $0 < r < a$ with zero boundary condition $u|_{r=a} = 0$ and initial value $u|_{t=0} = f(r, \theta)$. Again, we must perform Fourier - Bessel expansion on (95) and (96). The solution is given by the expression:

$$U(r, \theta, t) = \sum_{N=1}^{\infty} \sum_{n=-\infty}^{\infty} f_{nN} e^{-in\theta} J_n\left(\frac{r}{a} a_N^n\right) e^{-\kappa(\frac{a_N}{a})^2 t} \tag{102}$$

The Green function of the the diffusion equation on the circle can be written as follows:

$$G(r, r_0, \theta - \theta_0, \tau) = \frac{r_0}{2\pi} \sum_{N=1}^{\infty} \sum_{n=-\infty}^{\infty} \frac{1}{\lambda_{nN}^2} e^{i(\theta-\theta_0)} J_n\left(\frac{r}{a} a_N^n\right) J_n\left(\frac{r_0}{a} a_N^n\right) \frac{1}{\sqrt{4k\kappa\tau}} e^{-\kappa(\frac{a_N}{a})^2 \tau} \tag{103}$$

A solution of the uniform diffusion equation inside the circle

$$\frac{\partial U}{\partial t} = \kappa \Delta U + G(r, \theta, \tau) \tag{104}$$

$$U|_{t=0} = f(r, \theta), \quad U|_{r=a} = 0$$

is expressed through this Green function by the use of the standard formula

$$U(r, \theta, t) = \int_0^{2\pi} d\phi_0 \int_0^a f(r_0, \theta_0) G(r, r_0, \theta - \theta_0, t) dr_0 \tag{105}$$

$$+ \int_0^t d\tau \int_0^{2\pi} d\phi_0 \int_0^a g(r_0, \theta_0, \tau) G(r, r_0, \theta - \theta_0, t - \tau) dr_0$$

The solution of the forced wave equation

$$\frac{\partial^2 U}{\partial t^2} = c^2 \Delta U + g(r, \theta, t)$$

$$U|_{t=0} = A(r, \theta) \; U_t|_{t=0} = B(r, \theta)$$

is presented through the Green function:

$$G(r,r_0,\theta-\theta_0,\tau)=\frac{r_0}{2\pi}\sum_{N=1}^{\infty}\sum_{n=-\infty}^{\infty}\frac{1}{\lambda_{nN}^2 w_{nN}}e^{i(\theta-\theta_0)}J_n\left(\frac{r}{a}a_N^n\right)J\left(\frac{r_0}{a}a_N\right)\sin w_{nN} \quad (106)$$

by the standard formula again:

$$U(r,\theta,t)=\int_0^{2\pi}d\theta_0\int_0^a A(r_0,\theta_0)G_t(r,r_0,\theta-\theta_0,t)dr_0+ \quad (107)$$

$$+\int_0^{2\pi}d\theta_0\int_0^a B(r_0,\theta_0)G(r,r_0,\theta-\theta_0,t)dr_0+$$

$$\int_0^t d\tau\int_0^{2\pi}d\theta_0\int_0^a g(r_0,\theta_0,\tau)G(r,r_0,\theta-\theta_0,t-\tau)dr_0$$

Where w_{nN} are eigenfrequencies ($w_{nN}=\frac{c}{a}a_N^n$). Using Table 1, one can order the eigenfrequencies as follows.

$$w_{q+1}>w_q, \quad q_1=1,\cdots,\infty$$

$$w_1=w_0a_1^0=2.4048w_0, \quad w_0=\frac{c}{a}$$

$$w_2=w_0a_1^1=3.8317w_0$$

$$w_3=w_0a_1^2=5.1336w_0$$

$$w_4=w_0a_2^0=5.5201w_0$$

$$w_5=w_0a_3^0=6.3802w_0$$

If the Dirichlet boundary condition is replaced by the Neumann boundary condition. In this case instead of $J_n(\frac{r}{a}a_N)$ there should be $J_n(\frac{r}{a}b_N^n)$. We must now correct the value for λ_{nm}^2:

$$\lambda_{nm}^2=\int_0^a J_n^2\left(\frac{r}{a}b_N^n\right)rdr=\frac{a^2}{b_N^2}\int_0^{b_N}J_n^2(z)zdz \quad (108)$$

10 Bessel Functions of Second and Third Kinds

Bessel functions of the second kind, also called Neumann or Weber's Functions, are defined as solutions of the Bessel equation with the following asymptotics:

$$N_s(z) \to \sqrt{\frac{2}{\pi z}} \sin\Big(z - \frac{\pi s}{2} - \frac{\pi}{4}\Big) \tag{109}$$

If s is not an integer, one can present $N_s(z)$ as a linear combination of J_s and J_{-s}

$$N_s(z) = AJ_s + BJ_{-s} \tag{110}$$

To find A, B we have to solve the following equation:

$$\begin{aligned} \sin(\phi) \quad &= A\cos(\phi) + B\cos(\phi + \pi s) \\ \phi \quad &= z - \frac{\pi s}{2} - \frac{\pi}{4} \end{aligned}$$

As far as

$$\cos(\phi + \pi s) = \cos(\phi)\cos(\pi s) - \sin(\phi)\sin(\pi s)$$

we get

$$\begin{aligned} &A + B\cos(\pi s) = 0 \\ &1 = -B\sin(\pi s) \end{aligned}$$

Finally,

$$N_s = \frac{\cos(\pi s)J_s - J_{-s}}{\sin(\pi s)} \tag{111}$$

What is going on if $s \to n$? As far as $cos(\pi n) = (-1)^n$, $J_{-n} = (-1)^n J_n$, both the numerator and denominator in (111) tend to zero and we should use L'hopital's Rule. As a result:

$$N_s(z) = \frac{\cos(\pi n)\frac{\partial}{\partial s}J_s - \frac{\partial}{\partial s}J_{-s}\mid_{s=n}}{\pi\cos(\pi n)} = \frac{1}{\pi}\Big[\frac{\partial}{\partial s}J_s - (-1)^n\frac{\partial}{\partial s}J_{-s}\mid_{s=-n}\Big] \tag{112}$$

To study the behavior of $N_n(z)$ at $z \to 0$ one can use the series representation of (29)

$$J_s(z) = \left(\frac{z}{2}\right)^s \sum_{k=0}^{n} \frac{(-1)^k}{k!\Gamma(s+k+1)} \left(\frac{z}{2}\right)^k$$

Now,

$$\left(\frac{z}{2}\right) = e^{s \cdot \ln \frac{z}{2}}$$

$$\frac{d}{ds}\left(\frac{z}{2}\right)^s = \ln\left(\frac{z}{2}\right) e^{s \cdot \ln \frac{z}{2}} = \ln\left(\frac{z}{2}\right)\left(\frac{z}{2}\right)^s$$

In the same way,

Line is Cut Off in Notes

Thus, $N_n(z)$ can be presented as follows:

$$\pi N_n(z) = 2 \ln\left(\frac{z}{2}\right) J_n(z) - \widetilde{N}_n(z) \tag{113}$$

In (113), $\widetilde{N}_n(z) = N_n^{(1)}(z) + N_n^{(2)}(z)$ appears as a result of differentiating $\frac{1}{\Gamma(\pm s+k+1)}$ by s. Finally, we put $s = n$, then $\widetilde{N}_n(z)$ is expanision only over integer powers of n, both positive and negative.

When $s \to -n$, the first n terms in the expansion (29) tend to zero. However, their derivatives by s are not zero. They can easily be calculated. The last term in J_s vanishes as $s \to -n$, which corresponds to $k = n - 1$. This term gives the following contribution to $J_{-s}(z)$

$$\frac{1}{(n-1)!\Gamma(-s+n)} \left(\frac{z}{2}\right)^{n-2} = \frac{-s+n}{(n-1)!\Gamma(-s+n+1)} \left(\frac{z}{2}\right)^{n-2}$$

Differentiating the above expression by s at $s = n$ gives the following contribution to $N_n^{(1)}(z)$:

$$\frac{1}{(n-1)!} \left(\frac{z}{2}\right)^{n-2}$$

Collecting all similar terms ($k \leq n-1$) together, we end up with the following explicit expression for $N_n^{(1)}(z)$

$$N_n^{(1)}(z) = \sum_{k=0}^{n-1} \frac{(n-k-1)!}{k!} \left(\frac{z}{2}\right)^{2k-n} \tag{114}$$

$N_n^{(1)}(z)$ has singularities as $z \to \infty$. The most singular term is:

$$N_n^{(1)}(z) \to (n-1)! \left(\frac{2}{z}\right)^n \tag{115}$$

$$N_n^{(2)}(z) = \sum_{k=0}^{\infty} a_k (-1)^k \left(\frac{z}{2}\right)^{2k+n} \tag{116}$$

is the fast converging power series in (116)

$$a_k = \frac{1}{k!(n+k)!} \Big[\psi(k+1) + \psi(k+n+1)\Big] \tag{117}$$

Here, $\psi(s)$ is a new special function

$$\psi(s) = \frac{d}{ds} \ln(\Gamma(s)) = \frac{\Gamma'(s)}{\Gamma(s)}$$

$$\Gamma'(s) = \int_0^\infty e^{-t} \ln(t) t^{s-1} dt \tag{118}$$

Through the use of integration by parts, one can find the explicit value of $\psi(s)$ in all integral points can be [Line Cut Off Notes]. Remembering that $e^{-t} = -\frac{d}{dt} e^{-t}$, we get from (118)

$$\Gamma'(s) = (s-1)\Gamma'(s-1) + \Gamma(s-1)$$

By dividing by $\Gamma(s)$ and replacing $s \to s = 1$, we end up with the difference equation for ψ

$$\psi(s+1) = \psi(s) + \frac{1}{s}$$

$$\psi(1) = \int_0^\infty e^{-t} \ln(t) dt = -c \qquad (c \approx 0.5772) \tag{119}$$

c is the so-called Euler constant. Finally, we get for any integral point.

$$\psi(k+1) = -c + 1 + \cdots + \frac{1}{k} \tag{120}$$

To finish with the Neumann functions, we present the asymptotic behavior of the first two at $z \to 0$

$$N_0(z) \to 2\Big(\ln\frac{z}{2} + c\Big) + 0(z^2)$$

$$N_1(z) \to -\frac{2}{z} + 0(z)\cdot\ln(z) \tag{121}$$

Functions N_n are even if n is even and odd if N_n is odd.

$$N_n(-z) = (-1)^n N_n(z) \tag{122}$$

The Bessel functions of third kind are also know as the Hankel functions, which are defined as follow:

$$H_s^{(1)}(z) = J_s(z) + iN_s(z)$$

$$H_s^{(2)}(z) = J_s(z) - iN_s(z) \tag{123}$$

At $z \to \infty$ they have the following amymptotics

$$H_s^{(1)}(z) \to \sqrt{\frac{2}{\pi z}} e^{i\left(z - \frac{\pi s}{2} - \frac{\pi}{4}\right)}$$

$$H_s^{(2)}(z) \to \sqrt{\frac{2}{\pi z}} e^{-i\left(z - \frac{\pi s}{2} - \frac{\pi}{4}\right)} \tag{124}$$

apparently, $H_s^{(2)}(z) = \bar{H}_s^{(1)}(z)$. The Neumann functions are plotted in Figure 2.

11 Modified Bessel Functions

Modified Bessel functions are solutions of the modified Bessel equation

$$\frac{1}{z}\frac{d}{dz}z\frac{dR}{dz} - \Big(1 - \frac{s^2}{z^2}\Big)R = 0 \tag{125}$$

This equation appears if we perform the transformation $z \to iz$. In other words, the modified Bessel functions are Bessel functions of imaginary argument. However, one muse be careful performing the transform $z \to iz$ because we have to observe not only asymptotics at $z \to 0$, but also asmyptotics at $z \to \infty$. One solution $Y_s(z)$ of equation (125) is defined by the series

$$I_s(z) = \left(\frac{z}{2}\right)^s \sum_{k=0}^{\infty} \frac{1}{k!\Gamma(k+s+1)} \left(\frac{z}{2}\right)^{2k} \tag{126}$$

At $z \to 0$

$$I_s(z) \approx \frac{z^s}{z^s \Gamma(s+1)} \tag{127}$$

At $z \to \infty$

$$I_s(z) \to \sqrt{\frac{2}{\pi z}} e^z \tag{128}$$

I_s is the real function of real argument. They are connected with Bessel functions of the first kind by the relation:

$$I_s(z) = e^{-\frac{\pi}{2} si} J_s(iz) \tag{129}$$

In particular,

$$I_n(z) = -i^n J_n(iz) \tag{130}$$

Modified Bessel functions of second kind are defined by the relation

$$k_s(z) = \frac{\pi i}{2} e^{\frac{\pi}{2} si} H_s^{(1)}(iz) \tag{131}$$

They have asymptotics at

$$k_s(z) \approx \sqrt{\frac{\pi}{2z}} e^{-z}, \qquad z \to \infty \tag{132}$$

Both $I_s(z)$ and $k_s(z)$.

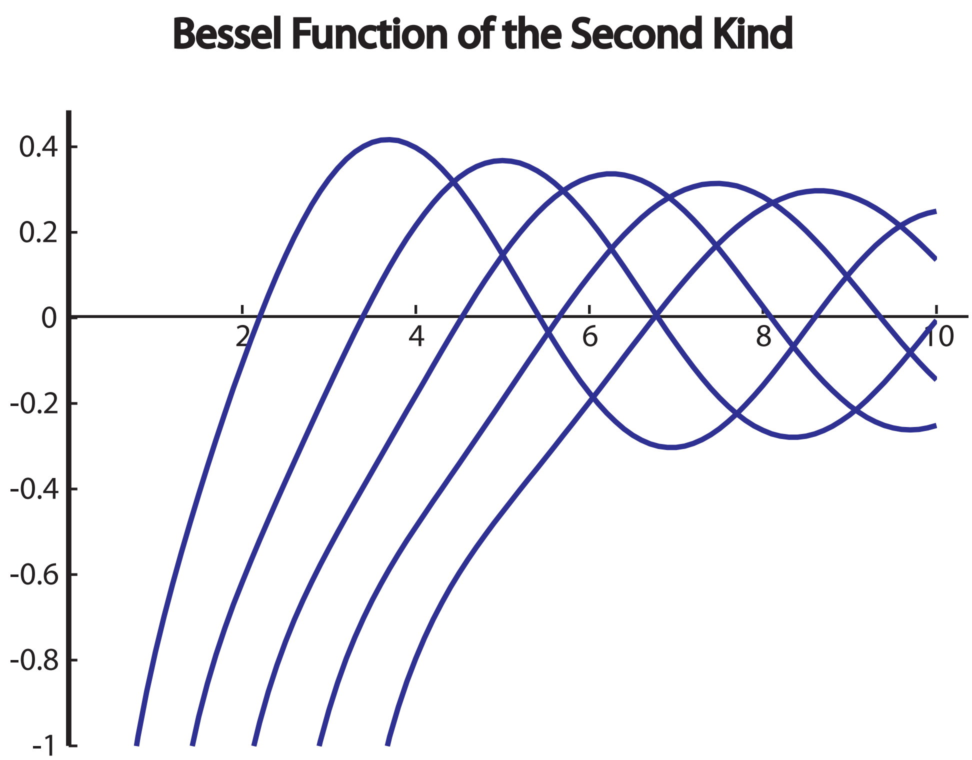
Modified Bessel functions of the First and Second kind are plotted on Figures 3 and 4.

12 Applications of the Modified Bessel Function

The modified Bessel functions are commonly used for solutions to many different applied problems. Let us consider the cylinder of radius R and length $2a$

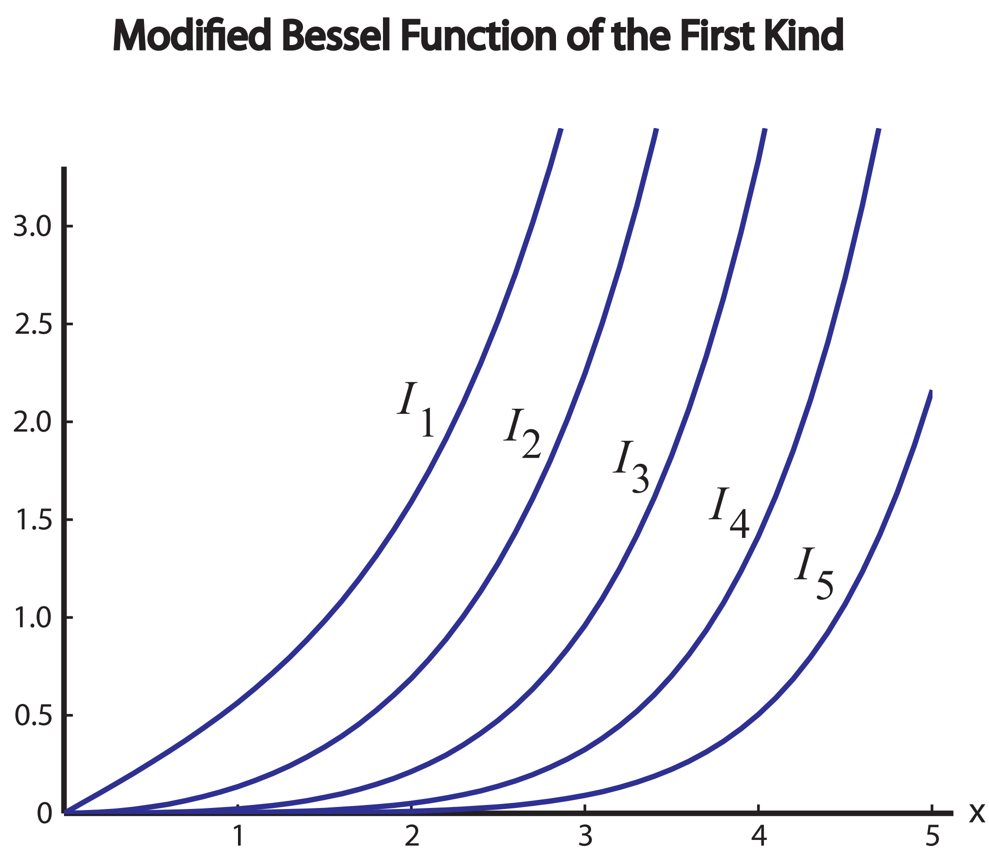
$$0 < r < R \qquad -a < z < a$$

Figure 2



A Bessel function of the second kind Nn(x) is a solution to the BESSEL DIFFERENTIAL EQUATION which is singular at the origin. Bessel functions of the second kind are also called NEUMANN FUNCTIONS or WEBER FUNCTIONS. The above plot shows Nn(x) for n = 1 , 2 , ... , 5

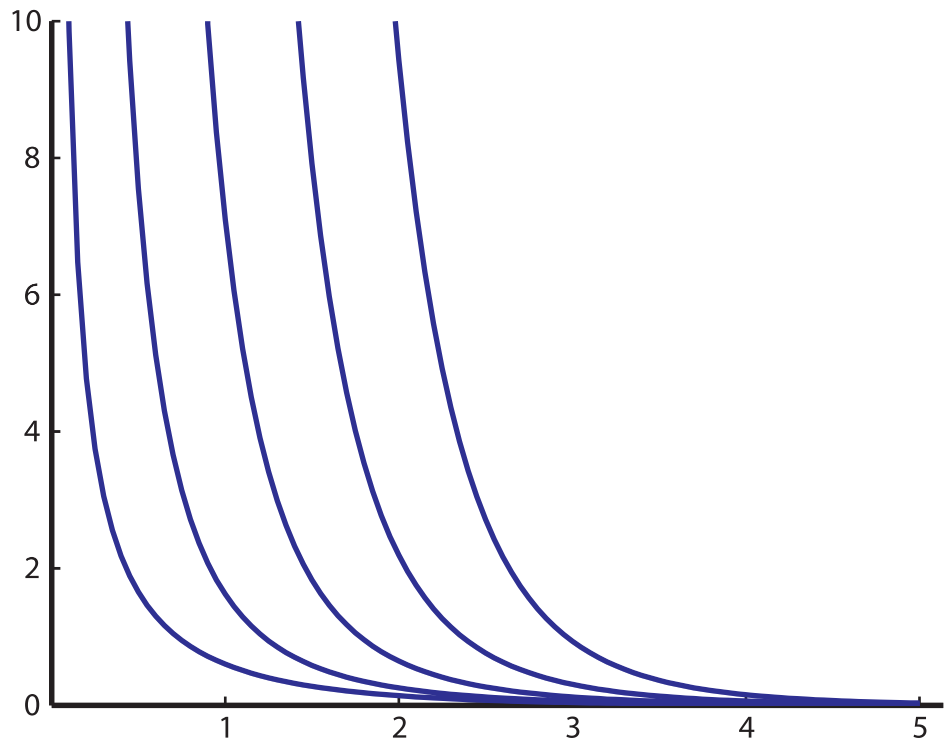
Figure 3



A function In(x) which is one of the solutions to the MODIFIED BESSEL DIFFERENTIAL EQUATION and is closely related to the BESSEL FUNCTION OF THE FIRST KIND Jn(x). The above plot In(x) for n = 1, 2, ... , 5.

Figure 4

Modified Bessel Function of the Second Kind



The function Kn(x) which is one of the solutions to the MODIFIED BESSEL DIFFERENTIAL EQUATION. The above plot shows Kn(x) for n = 1, 2, ... , 5.

Let us solve the Laplace equation:

$$\Delta u = \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} = 0 \tag{133}$$

with the "lateral" boundary conditions:

$$u|_{z=a} = 0, \quad u|_{z=-a} = 0$$

$$u|_{r=R} = f(z,\theta) \qquad -a < z < a$$

$$f(z, \theta + 2\pi) = f(z,\theta)$$

To solve equation (133), we will use separation of variables and look for solutions in the form:

$$u = e^{in\theta}\cos\left(\frac{\pi m}{2a}z\right)R_{n,m}(r) \tag{134}$$

$R_{n,m}$ satisfies the following equation:

$$\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial R}{\partial r} - \left(k_m^2 + \frac{n^2}{z^2}\right)R = 0, \quad k_m^2 = \frac{\pi m}{2a} \tag{135}$$

Equation (135) has the following solution satisfying the condition:

$$R|_{r=R} = 1, \quad R = \frac{I_n(k_m z)}{I_n(k_m R)} \tag{136}$$

The solution of the Laplace equation is:

$$u = \sum_{n=-\infty}^{\infty}\sum_{m=1}^{\infty} f_{n,m} e^{in\theta}\cos\left(\frac{\pi m}{2a}z\right)\frac{I_n(k_m z)}{I_n(k_m R)} \tag{137}$$

$f_{n,m}$ - coefficient of double Fourier series to be found from the condition:

$$f(z,\theta) = \sum_{n=-\infty}^{\infty}\sum_{m=1}^{\infty} f_{n,m} e^{in\theta}\cos\left(\frac{\pi m}{2a}z\right) \tag{138}$$

They are:

$$f_{n,m} = \frac{1}{2\pi a}\int_0^{2\pi} d\theta \int_{-a}^{a} f(\theta, z) e^{-in\theta}\cos\left(\frac{\pi m}{2a}z\right)dz \tag{139}$$

Let us now solve the Laplace equation (133) outside the infinite cylinder ($0 < r < R$ and $-\infty < z < \infty$). Again, the boundary condition is:

$$u|_{r=R} = f(z,\theta) \qquad -\infty < z < \infty$$

$$f(z,\theta+2\pi) = f(z,\theta)$$

To solve the problem, we must specify the boundary condition in infinity:

$$\frac{\partial u}{\partial r} \to 0 \quad at \quad r \to \infty$$

Now, we should perform the Fourier transformations in the z-direction:

$$u(k,\theta,r) = \int_{-\infty}^{\infty} u(z,\theta,r)e^{-ikz}dz$$

$$f(k,\theta) = \int_{-\infty}^{\infty} f(z,\theta)e^{-ikz}dz \tag{140}$$

and realize the expansion in the Fourier series in angles, thus:

$$f(k,\theta) = \sum f_n(k)e^{in\theta}$$

$$f_n(k) = \frac{1}{2\pi}\int_0^{2\pi} f(k,\theta)e^{-in\theta}d\theta \tag{141}$$

The radial part of the solution satisfies the modified Bessel equation:

$$\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial R}{\partial r} - \left(k^2 + \frac{n}{r}\right)R = 0 \tag{142}$$

It should be taken as follows:

$$R(k,n) = \frac{K_n(kr)}{K_n(kR)}$$

The solution is given by the inverse Fourier Transform:

$$u(z,r,\theta) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dt \sum_{n=-\infty}^{\infty} f_n(k)\frac{K_n(kr)}{K_n(kR)}e^{-ikz+in\theta} \tag{143}$$

13 Heavy Chain

The future Bessel function appeared in mathematics in 1732 when Danie Bernoully solved the problem on oscillations of the hung, heavy chain. Let the chain of length l and linear density ρ be hung such that it can move only in one direction. Let the coordinate x be taken such that $x = 0$ at the free end of the chain. Then the deviation from equilibrium state $u = u(x,t)$ satisfies the equation:

$$\frac{\partial^2 u}{\partial t^2} = g\Big(x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x}\Big) \tag{144}$$

$$u|_{x=l} = 0$$

Use separation of variables:

$$u = X(x)T(t)$$

where $T(t) = \sin(\omega t + \varphi)$ leads to the equation:

$$xX'' + X'(x) + \frac{\omega^2}{g}X(x) = 0 \tag{145}$$

with the boundary condition:

$$X(l) = 0, \quad X(0) < \infty$$

By introducing the new variable:

$$y = 2\omega\sqrt{\frac{x}{g}}$$

We transform (145) to the Bessel equation:

$$\frac{d^2X}{dy^2} + \frac{1}{y}\frac{dX}{dy} + X = 0 \tag{146}$$

This is the equation for Bessel functions of zero order. Thus, the solution is:

$$u = AJ_0\left(2\omega\sqrt{\frac{x}{g}}\right) \tag{147}$$

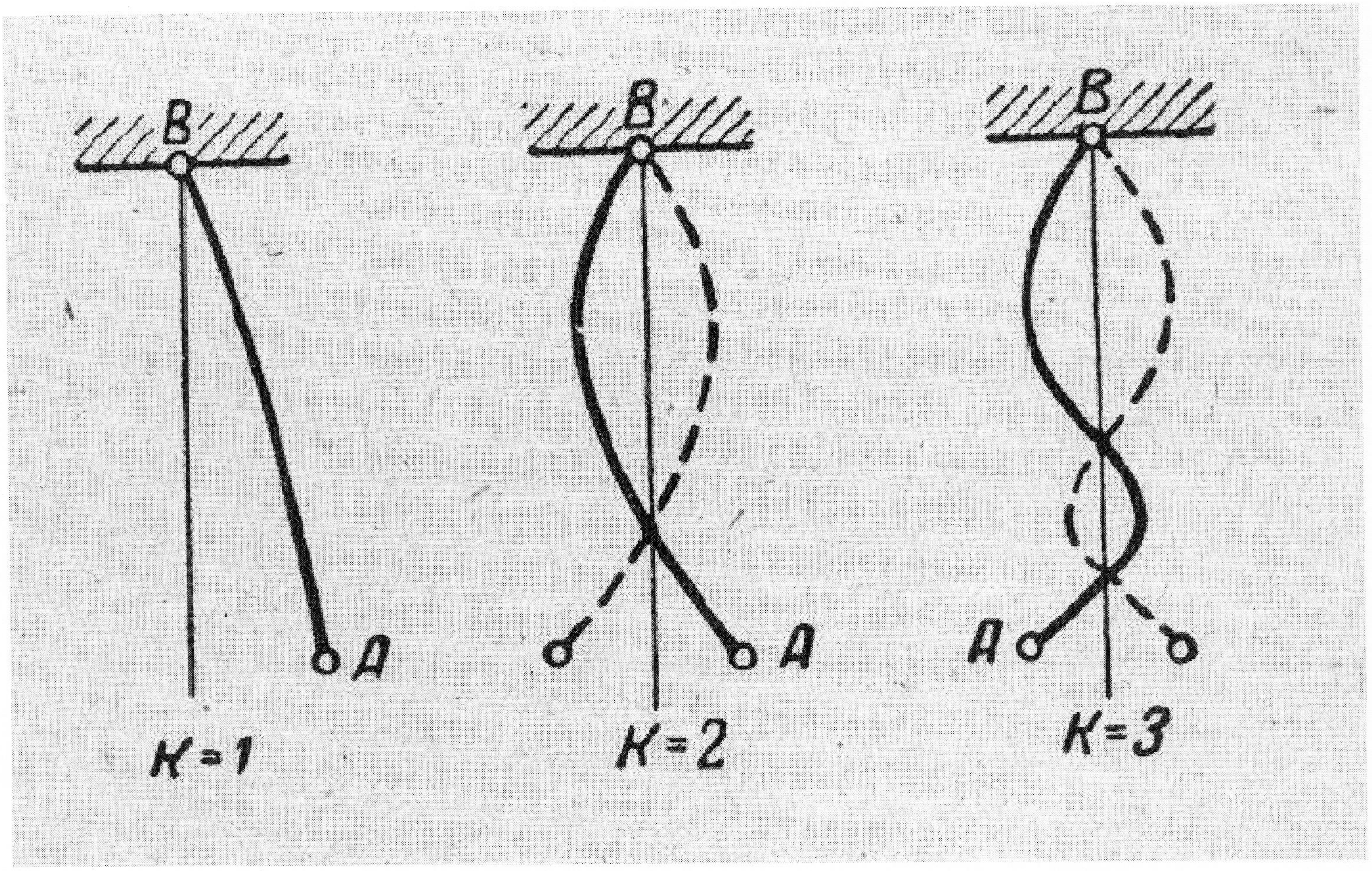
The characteristic frequency ω_k can take consequence of discrete values $(\omega_k, k = 1, 2, \cdots, \infty)$. They can be found from the boundary condition:

$$u(l) = 0 \qquad J_0\left(2\omega_k\sqrt{\frac{l}{y}}\right) = 0$$

Hence,

$$2\omega_k\sqrt{\frac{l}{g}} = a_k^0 \qquad J_0(a_k^0) = 0$$
$$\omega_k = \frac{1}{2}\sqrt{\frac{g}{l}}a_k^0$$

a_n^0 - zeros of the Bessel function J_0.


B
A
K=1
B
A
K=2
B
A
K=3